

Large p -groups actions with a p -elementary abelian second ramification group.

Magali Rocher.

Abstract

Let k be an algebraically closed field of characteristic $p > 0$ and C a connected nonsingular projective curve over k with genus $g \geq 2$. Let (C, G) be a "big action", i.e. a pair (C, G) where G is a p -subgroup of the k -automorphism group of C such that $\frac{|G|}{g} > \frac{2p}{p-1}$. We denote by G_2 the second ramification group of G at the unique ramification point of the cover $C \rightarrow C/G$. The aim of this paper is to describe the big actions whose G_2 is p -elementary abelian. In particular, we obtain a structure theorem by considering the k -algebra generated by the additive polynomials. We more specifically explore the case where there is a maximal number of jumps in the ramification filtration of G_2 . In this case, we display some universal families.

1 Introduction.

Setting. Let k be an algebraically closed field of characteristic $p > 0$. We denote by C a connected nonsingular projective curve over k , with genus $g \geq 2$, and by G a p -subgroup of the k -automorphism group of C : $\text{Aut}_k(C)$, such that $\frac{|G|}{g} > \frac{2p}{p-1}$. Such a pair (C, G) is called a "big action". Then, there is a point of C (say ∞) such that G is equal to the wild inertia subgroup G_1 of G at ∞ (cf. [LM05]). Moreover, the quotient curve C/G is isomorphic to the projective line \mathbb{P}_k^1 and the ramification locus (respectively branch locus) of the cover $\pi : C \rightarrow C/G$ is the point ∞ (respectively $\pi(\infty)$). Furthermore, the second lower ramification group G_2 of G at ∞ is non trivial and is strictly included in G_1 . In addition, the quotient curve C/G_2 is isomorphic to \mathbb{P}_k^1 and the quotient group G/G_2 acts as a group of translations: $\{X \rightarrow X + y, y \in V\}$ of the affine line $C/G_2 - \{\infty\}$.

Motivation and purpose. When searching for a classification of big actions, it naturally occurs that the quotient $\frac{|G|}{g^2}$ has a "sieve" effect. Stichtenoth shows that, for any p -subgroup G of $\text{Aut}_k(C)$, $\frac{|G|}{g^2} \leq \frac{4p}{(p-1)^2}$ (cf. [St73]). Later on, Lehr and Matignon prove that the big actions such that $\frac{|G|}{g^2} \geq \frac{4}{(p-1)^2}$ correspond to the p -cyclic étale covers of the affine line given by an Artin-Schreier equation: $W^p - W = f(X) := X S(X) + c X \in k[X]$, where $S(X)$ runs over the additive polynomials of $k[X]$ (cf. [LM05]). In a sequel paper [Ro3], we go further in the classification and describe the big actions such that $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$. Under this condition, it is shown in [MR08] that G_2 is a p -elementary abelian group of order dividing p^3 , hence the necessity to study big actions whose second ramification group G_2 is p -elementary abelian.

Outline of the paper. The main result of this paper is a structure theorem for the big actions with a p -elementary abelian second ramification group G_2 . This result is obtained by considering the k -algebra generated by the additive polynomials of $k[X]$. Indeed, let (C, G) be a big action whose G_2 is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. Then, the function field of the curve is parametrized by n Artin-Schreier equations: $W_i^p - W_i = f_i(X) \in k[X]$, with $1 \leq i \leq n$. For all $t \geq 1$, we define Σ_t as the k -subvector space of $k[X]$ generated by 1 and the products of at most t additive polynomials of $k[X]$. In section 3, we prove that each function f_i belongs to Σ_{i+1} , which means that we can express f_i as a linear combination over k of products of at most $i+1$ additive polynomials of $k[X]$. This result generalizes the p -cyclic case, i.e. $n = 1$, but, contrary to this case, the converse is no longer true, which means that such a family $(f_i)_{1 \leq i \leq n}$ does not necessarily give birth to a big action, except under specific conditions that are studied in what follows. The obstruction essentially lies in the embedding problem associated with the exact sequence:

$$0 \longrightarrow G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n \longrightarrow G \longrightarrow V \longrightarrow 0$$

More precisely, we study the induced representation $\phi: G/G_2 \rightarrow \text{Aut}(G_2) \simeq \text{Gl}_n(\mathbb{F}_p)$ via the representation dual with respect to the Artin-Schreier pairing (see section 2).

Sections 4 and 5 are devoted to two special cases of main interest. In section 4, we investigate the case where there is only one jump in the upper ramification filtration of G_2 . Then, the representation mentioned above is trivial or, equivalently, each function f_i belongs to Σ_2 . In section 5, we give a group-theoretic characterization of what can be regarded as the opposite case, namely: $f_i \in \Sigma_{i+1} - \Sigma_i$. Then, there is a maximal number of jumps in the upper ramification filtration of G_2 . This case is relevant insofar as the representation ϕ is non trivial and provides much information. To conclude, section 6 is devoted to examples illustrating section 5. In particular, we display a universal family parametrizing the big actions (C, G) that satisfy $f_i \in \Sigma_{i+1} - \Sigma_i$, for $p = 5$, a given $n \leq p - 1$ and $\dim_{\mathbb{F}_p} V = 2$. When investigating the properties of the corresponding group G , we show that the center of G is cyclic of order p and relate G with capable groups as defined by Hall (cf. [Ha40]).

Notation and preliminary remarks. Let k be an algebraically closed field of characteristic $p > 0$. We denote by F the Frobenius endomorphism for a k -algebra. Then, \wp means the Frobenius operator minus identity. We denote by $k\{F\}$ the k -subspace of $k[X]$ generated by the polynomials $F^i(X)$, with $i \in \mathbb{N}$. It is a ring under the composition. Furthermore, for all α in k , $F\alpha = \alpha^p F$. The elements of $k\{F\}$ are the additive polynomials, i.e. the polynomials $P(X)$ of $k[X]$ such that for all α and β in k , $P(\alpha + \beta) = P(\alpha) + P(\beta)$. Moreover, a separable polynomial is additive if and only if the set of its roots is a subgroup of k (see [Go96] chap. 1).

Let $f(X)$ be a polynomial of $k[X]$. Then, there is a unique polynomial $\text{red}(f)(X)$ in $k[X]$, called the reduced representative of f , which is p -power free, i.e. $\text{red}(f)(X) \in \bigoplus_{(i,p)=1} k X^i$, and such that $\text{red}(f)(X) = f(X) \bmod \wp(k[X])$. We say that the polynomial f is reduced mod $\wp(k[X])$ if and only if it coincides with its reduced representative $\text{red}(f)$. The equation $W^p - W = f(X)$ defines a p -cyclic étale cover of the affine line that we denote by C_f . Conversely, any p -cyclic étale cover of the affine line $\text{Spec } k[X]$ corresponds to a curve C_f where f is a polynomial of $k[X]$ (see [Mi80] III.4.12, p. 127). By Artin-Schreier theory, the covers C_f and $C_{\text{red}(f)}$ define the same p -cyclic covers of the affine line. The curve C_f is irreducible if and only if $\text{red}(f) \neq 0$.

Throughout the text, the pair (C, G) denotes a big action such that the second ramification group G_2 is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. We denote by $L := k(C)$ the function field of C and by $k(X) := L^{G_2}$ the subfield of L fixed by G_2 . The extension L/L^{G_2} is an étale cover of the affine line $\text{Spec } k[X]$ whose Galois group is $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$. Therefore, it can be parametrized by n Artin-Schreier equations: $W^p - W = g_i(X)$, with $1 \leq i \leq n$. In other words, $L = k(X, W_1, \dots, W_n)$. As seen above, the functions $g_i(X)$ can be chosen in $k[X]$. Moreover, the quotient group G/G_2 is a group of automorphisms of $k[X]$. Since it is a p -group, it actually acts as a group of translations of $\text{Spec } k[X]$, through $\tau_y : X \rightarrow X + y$, where y runs over a subgroup V of k . We remark that V is an \mathbb{F}_p -subvector space of k . We denote by v its dimension and thus obtain the exact sequence:

$$0 \longrightarrow G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n \longrightarrow G = G_1 \xrightarrow{\pi} V \simeq (\mathbb{Z}/p\mathbb{Z})^v \longrightarrow 0$$

where for all g in G , $\pi(g) := g(X) - X$. We also fix a set theoretical section, i.e. a map $s : V \rightarrow G$, such that $\pi \circ s = \text{id}_V$.

2 An embedding problem.

2.1 An \mathbb{F}_p -vector space dual of G_2 .

We first exhibit an \mathbb{F}_p -vector space dual of G_2 . Following Artin-Schreier theory (see [Bo83], chap. IX, ex. 19), we define the \mathbb{F}_p -vector space:

$$\tilde{A} := \frac{\wp(L) \cap k(X)}{\wp(k(X))}$$

In other words, \tilde{A} is the \mathbb{F}_p -vector space generated by the classes of the functions $g_i(X)$ modulo $\wp(k(X))$. The inclusion $k[X] \subset k(X)$ induces an injection:

$$A := \frac{\wp(L) \cap k[X]}{\wp(k[X])} \hookrightarrow \tilde{A}$$

Since the extension is étale outside ∞ , the functions $g_i(X)$ parametrizing the extension $L/k(X)$ can be chosen in $k[X]$. It follows that we can identify A with \tilde{A} . Consider the Artin-Schreier pairing:

$$\begin{cases} G_2 \times A \longrightarrow \mathbb{Z}/p\mathbb{Z} \\ (g, \overline{\wp w}) \longrightarrow [g, \overline{\wp w}] := g(w) - w \end{cases}$$

where g belongs to $G_2 \subset \text{Aut}_k(L)$, w is an element of L such that $\wp w \in k[X]$ and $\overline{\wp w}$ denotes the class of $\wp w \bmod \wp(k[X])$. This pairing is non degenerate, which implies that, as an \mathbb{F}_p -vector space, A is dual to G_2 .

2.2 Two dual representations.

We now introduce two representations dual with respect to the Artin-Schreier pairing. The first representation, say ϕ , expresses the action of G_1 on G_2 via conjugation. Indeed, for all y in V , we define an automorphism $\phi(y)$ of G_2 such that, for all g in G_2 , $\phi(y)(g) := s(y)^{-1} g s(y)$. Since G_2 is abelian, $\phi(y)$ does not depend on the lifting $s(y)$ in G_1 chosen for y in V . Therefore, there exists a representation ϕ which maps each y in V to $\phi(y)$ in $\text{Aut}(G_2)$.

Then, we display a second representation expressing the action of V on A . More precisely, for all y in V , we consider the automorphism $\rho(y)$ of A defined as follows:

$$\rho(y) : \begin{cases} A \rightarrow A \\ \overline{\wp w} \rightarrow \overline{\wp(s(y)(w))} \end{cases}$$

where w is an element of L such that $\wp w \in k[X]$. As $s(y)$ belongs to $G_1 \subset \text{Aut}_k(L)$, then $s(y)(w)$ still lies in L . Furthermore, since $w^p - w \in k[X]$, then $(s(y)(w))^p - s(y)(w) = s(y)(w^p - w) \in s(y)(k[X]) \subset k[X]$, as $s(y)(X) = X + y$. This ensures that $\rho(y)$ is well-defined. Moreover, as G_2 trivially acts on $A \subset k(X) = L^{G_2}$, $\rho(y)$ is independent of the lifting $s(y) \in G_1$ chosen for y . Accordingly, we can define a representation ρ which maps each y in V to $\rho(y)$ in $\text{Aut}(A)$.

Remark 2.1. Note that for all $\overline{f(X)}$ in A and for all y in V , $\rho(y)\overline{f(X)} = \overline{f(X+y)}$.

Proposition 2.2. The two representations ρ and ϕ , as defined above, are dual with respect to the Artin-Schreier pairing.

Proof: For all y in V , for all g in G_2 and for all w in L such that $\wp w$ is in $k[X]$,

$$\begin{aligned} [\phi(y)(g), \overline{\wp w}] &= [s(y)^{-1} g s(y), \overline{\wp w}] \\ &= s(y)^{-1} g s(y)(w) - w = s(y)^{-1} g s(y)(w) - s(y)^{-1} s(y)(w) \\ &= s(y)^{-1} (g s(y)(w) - s(y)(w)) = g s(y)(w) - s(y)(w) \end{aligned}$$

since $g s(y)(w) - s(y)(w) = [g, \overline{\wp(s(y)(w))}] \in \mathbb{F}_p$.

As a conclusion, $[\phi(y)(g), \overline{\wp w}] = [g, \overline{\wp(s(y)(w))}] = [g, \rho(y)(\overline{\wp w})] \in \mathbb{F}_p$ \square

Since the image of ρ is a unipotent subgroup of $GL_n(\mathbb{F}_p)$, one can find a basis for the \mathbb{F}_p -vector space A in which the image of the representation ρ can be identified with a subgroup of the upper triangular matrices in $GL_n(\mathbb{F}_p)$. A means to do so is to endow A with a filtration which proves to be dual of the upper ramification filtration of G_2 .

2.3 Dual filtrations on A and G_2 .

The following three subsections are classical. Nevertheless, it is more convenient to recall both the proofs and the construction so as to fix the notation.

2.3.1 A filtration and an adapted basis for A .

Definition 2.3. 1. We first gather from the canonical map "degree" a map defined on A in the following way:

$$\text{deg} : \begin{cases} A \rightarrow \mathbb{N} \cup \{-\infty\} \\ \overline{f(X)} \rightarrow \inf \{ \text{deg}(f + \wp(P)), P \in k[X] \} \end{cases}$$

2. For all i in \mathbb{N} , we define a sequence of \mathbb{F}_p -subvector spaces of A as follows:

$$A^i := \{ \overline{f(X)} \in A, \text{deg}(\overline{f(X)}) < i \}$$

3. From the increasing sequence: $\{0\} = A^0 \subset A^1 \subset A^2 \subset \dots \subset A^r \subset A^{r+1} = A$, we extract a strictly increasing sequence $(A^{\mu_i})_{0 \leq i \leq s}$ such that:

$$\{0\} = A^0 = \dots = A^{\mu_0} \subsetneq A^{\mu_0+1} = \dots = A^{\mu_1} \subsetneq A^{\mu_1+1} = \dots \subsetneq \dots \subsetneq A^{\mu_s} \subsetneq A^{\mu_s+1} = A$$

where the jumps μ_i are uniquely determined by the condition: $A^{\mu_i} \subsetneq A^{\mu_i+1}$. By definition of the function "degree" on A , all the integers μ_i are prime to p . By convenience of notation, we also define μ_{s+1} as $\mu_s + 1$ so that $A = A^{\mu_{s+1}}$. For all i in $\{0, \dots, s+1\}$, we denote by n_i the dimension of A^{μ_i} over \mathbb{F}_p . Note that $n_0 = 0$ and $n_{s+1} = n$.

4. Starting from a basis of A^{μ_1} , we complete it in a basis of A^{μ_2} , and so on until $A^{\mu_{s+1}}$. In this way, we construct a basis of A , say: $\{f_1(X), \dots, f_n(X)\}$, which is said to be "adapted" to the filtration defined above. Moreover, we impose specific conditions on the degree m_i of each $f_i(X)$:

- (a) $\forall i \in \{1, \dots, n\}$, m_i is prime to p .
- (b) $\forall i \in \{1, \dots, n-1\}$, $m_i \leq m_{i+1}$.
- (c) $\forall (\lambda_1, \dots, \lambda_n) \in \mathbb{F}_p^n$ not all zeros,

$$\deg \left(\sum_{i=1}^n \lambda_i \overline{f_i(X)} \right) = \max_{i=1, \dots, n} \{ \deg \lambda_i \overline{f_i(X)} \}.$$

Remark 2.4. Keeping the notation above, we notice that, for all $i \in \{0, \dots, s\}$, $m_{n_i+1} = m_{n_i+2} = \dots = m_{n_{i+1}} = \mu_i$.

This provides a new parametrization of the function field L . Indeed, for all i in $\{1, \dots, n\}$, we fix a representative mod $\wp(k[X])$ of $\overline{f_i(X)}$: $f_i(X)$ and assume it to be reduced mod $\wp(k[X])$. As m_i is prime to p , $f_i(X)$ still has degree m_i . We also suppose that for all i in $\{1, \dots, n\}$, $f_i(0) = 0$. From now on, the extension $L/k(X)$ is parametrized by the n Artin-Schreier equations: $W_i^p - W_i = f_i(X)$ with $1 \leq i \leq n$.

2.3.2 The link with the upper ramification filtration of G_2 .

In what follows, we highlight the correspondence between the jumps $(\mu_i)_{0 \leq i \leq s}$ in the filtration of A and the jumps $(\nu_i)_{0 \leq i \leq r}$ in the upper ramification filtration of G_2 . Since G_2 is abelian, the Hasse-Arf Theorem (see e.g. [Se68], Chap. IV) asserts that the jumps in the upper ramification filtration are integers. So the ramification filtration reads as follows:

$$G_2 = (G_2)^0 = \dots = (G_2)^{\nu_0} \supsetneq (G_2)^{\nu_0+1} = \dots = (G_2)^{\nu_1} \supsetneq \dots = (G_2)^{\nu_r} \supsetneq (G_2)^{\nu_r+1} = \{0\}$$

By convenience, put $\nu_{r+1} := \nu_r + 1$.

Proposition 2.5. Keeping the notation above, $r = s$ and for all i in $\{0, \dots, s+1\}$, $\mu_i = \nu_i$. It follows that the filtration of A and G_2 are dual with respect to the Artin-Schreier pairing, that is to say $(G_2)^{\nu_i}$ is the orthogonal of A^{μ_i} , for all i in $\{0, \dots, s+1\}$.

Proof: Let ν_i be a jump in the upper ramification filtration of G_2 , with $0 \leq i \leq r$. Since the $(G_2)^{\nu_i}$ are \mathbb{F}_p -subvector spaces of G_2 , one can find an index p -subgroup of G_2 , say \mathcal{H} , such that $(G_2)^{\nu_i+1} \subset \mathcal{H}$ and $(G_2)^{\nu_i} \not\subset \mathcal{H}$. As $L^{\mathcal{H}}/L^{G_2}$ is a p -cyclic cover of the affine line inside L , with Galois group equal to G_2/\mathcal{H} , it is parametrized by an Artin-Schreier equation: $W^p - W = f(X) = \sum_{i=1}^n \lambda_i f_i(X)$ with $(\lambda_i)_{1 \leq i \leq n} \in (\mathbb{F}_p)^n - \{(0, 0, \dots, 0)\}$. Condition (c) in Definition 2.3.4 requires: $\deg(f) = \max_{1 \leq i \leq n} \{ \deg \lambda_i f_i(X) \} \in \{m_i, 1 \leq i \leq n\} = \{\mu_i, 0 \leq i \leq s\}$. Besides, the group G_2 induces an upper ramification filtration on G_2/\mathcal{H} , namely $(\frac{G_2}{\mathcal{H}})^{\nu} = \frac{(G_2)^{\nu} \mathcal{H}}{\mathcal{H}}$ (see [Se68], Chap. IV, Prop. 14). Therefore, the ramification filtration of G_2/\mathcal{H} reads:

$$\mathbb{Z}/p\mathbb{Z} \simeq \frac{G_2}{\mathcal{H}} = (\frac{G_2}{\mathcal{H}})^0 = \dots = (\frac{G_2}{\mathcal{H}})^{\nu_i} \supsetneq (\frac{G_2}{\mathcal{H}})^{\nu_i+1} = \{0\}$$

This is precisely the p -cyclic case for which it is well-known that the only jump of ramification: ν_i is equal to $\deg(f)$ (see [Se68], Chap. IV, ex. 4, p. 80). Therefore, $\nu_i \in \{\mu_j, 0 \leq j \leq s\}$.

Conversely, consider μ_i , for $0 \leq i \leq s$. Then, by Remark 2.4, $\mu_i = m_{n_i+1}$, i.e. the degree of the function f_{n_i+1} . The function field of the curve: $W^p - W = f_{n_i+1}(X)$, is a p -cyclic étale cover of the affine line whose Galois group is an index p -subgroup of G_2 , say \mathcal{H} . We define the integer $\nu(G_2) \in \{\nu_i, 0 \leq i \leq r+1\}$ such that $G_2^{\nu(G_2)+1} \subset \mathcal{H}$ and $G_2^{\nu(G_2)} \not\subset \mathcal{H}$. As seen above, $m_{n_i+1} = \nu(G_2)$. Therefore, $\mu_i \in \{\nu_j, 0 \leq j \leq r\}$. Accordingly, $\{\nu_i, 0 \leq i \leq r\} = \{\mu_i, 0 \leq i \leq s\}$. They are both strictly increasing sequence, so $r = s$ and for all i in $\{0, \dots, s\}$, $\mu_i = \nu_i$. In addition, $\mu_{s+1} = \mu_s + 1 = \nu_r + 1 = \nu_{r+1}$, which completes the proof of the proposition. \square

2.3.3 The different exponent and the genus of the extension.

In this section, we establish a formula to calculate the different exponent and the genus of the extension L/L^{G_2} . We keep the notation defined in sections 2.3.1 and 2.3.2.

Proposition 2.6. *Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. The different exponent of the extension L/L^{G_2} is given by the formula:*

$$d = (p-1) \sum_{i=1}^n p^{i-1} (m_i + 1)$$

Proof: Since G_2 is abelian, one can apply to L/L^{G_2} the upper index version of the Hilbert's different formula as given in [Au00] (p. 120): $d = \sum_{i=0}^{\infty} (|G_2| - [G_2 : (G_2)^i])$. In our case, this formula reads:

$$d = (\nu_0 + 1) (|G_2| - [G_2 : (G_2)^{\nu_0}]) + \sum_{j=1}^r (\nu_j - \nu_{j-1}) (|G_2| - [G_2 : (G_2)^{\nu_j}])$$

Using Proposition 2.5, we obtain:

$$\begin{aligned} d &= (\nu_0 + 1) (|G_2| - |A^{\mu_0}|) + \sum_{j=1}^r (\nu_j - \nu_{j-1}) (|G_2| - |A^{\mu_j}|) \\ &= (\mu_0 + 1) (p^n - p^{n_0}) + \sum_{j=1}^s (\mu_j - \mu_{j-1}) (p^n - p^{n_j}) \\ &= \sum_{j=0}^s (p^n - p^{n_j}) (\mu_j + 1) + \sum_{j=1}^{s+1} (p^{n_j} - p^n) (\mu_{j-1} + 1) \\ &= \sum_{j=1}^{s+1} (p^n - p^{n_{j-1}}) (\mu_{j-1} + 1) + \sum_{j=1}^{s+1} (p^{n_j} - p^n) (\mu_{j-1} + 1) \\ &= \sum_{j=1}^{s+1} (p^{n_j} - p^{n_{j-1}}) (\mu_{j-1} + 1) \\ &= \sum_{i=1}^{s+1} \sum_{j=n_{i-1}+1}^{n_i} p^{j-1} (p-1) (\mu_{i-1} + 1) \\ &= (p-1) \sum_{i=1}^{s+1} \sum_{j=n_{i-1}+1}^{n_i} p^{j-1} (m_j + 1) \\ &= (p-1) \sum_{i=1}^n p^{i-1} (m_i + 1) \quad \square \end{aligned}$$

Note that another proof of this formula can be obtained by applying the formula given by Garcia and Stichtenoth in [GS91].

Corollary 2.7. *Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. The genus of the extension L/L^{G_2} is given by the formula:*

$$g = \frac{1}{2} (p-1) \sum_{i=1}^n p^{i-1} (m_i - 1)$$

Proof: The formula directly derives from the Hurwitz genus formula (see e.g. [St93]) combined with the formula given in Proposition 2.6:

$$2(g-1) = 2(g_{C/G_2} - |G_2|) + d = -2p^n + (p-1) \sum_{i=1}^n p^{i-1} (m_i + 1) \quad \square$$

2.4 Matricial representations of ρ and ϕ .

From now on, we work in the adapted basis constructed for A in section 2.3.1: $\{\overline{f_1(X)}, \dots, \overline{f_n(X)}\}$. For any y in V , we denote by $L(y)$ the matrix of the automorphism $\rho(y)$ in this basis. As indicated in Remark 2.1, we recall that for all y in V and for all i in $\{1, \dots, n\}$, $\rho(y) \overline{f_i(X)} = \overline{f_i(X+y)}$. Moreover, the conditions imposed on the degree of the functions $\overline{f_i(X)}$ imply that the matrix $L(y)$ belongs to $T_{1,n}^u(\mathbb{F}_p)$, the subgroup of $Gl_n(\mathbb{F}_p)$ made of the upper triangular matrices with identity on the diagonal. Thus, $L(y)$ reads as follows:

$$L(y) := \begin{pmatrix} 1 & \ell_{1,2}(y) & \ell_{1,3}(y) & \cdots & \ell_{1,n}(y) \\ 0 & 1 & \ell_{2,3}(y) & \cdots & \ell_{2,n}(y) \\ 0 & 0 & \cdots & \cdots & \ell_{i,n}(y) \\ 0 & 0 & 0 & 1 & \ell_{n-1,n}(y) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in Gl_n(\mathbb{F}_p)$$

In other words,

$$\begin{aligned} \forall y \in V, f_1(X+y) - f_1(X) &= 0 \pmod{\wp(k[X])} \\ \forall i \in \{2, \dots, n\}, \forall y \in V, f_i(X+y) - f_i(X) &= \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X) \pmod{\wp(k[X])} \end{aligned} \quad (1)$$

Remark 2.8. We still denote by m_i the degree of the function f_i . We observe that the degree of the left-hand side of (1) is at most $m_i - 1$. It follows that, whenever $m_i = m_j$, f_j does not occur in the right-hand side of (1), which means that $\ell_{j,i}$ is zero on V .

Proposition 2.9. Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 2$. We keep the notation defined above.

1. For all i in $\{1, \dots, n-1\}$, $\ell_{i,i+1}$ is a linear form from V to \mathbb{F}_p .
2. For all i in $\{1, \dots, n-1\}$, put $\mathcal{L}_{i,i+1}(X) := \prod_{y \in \text{Ker} \ell_{i,i+1}} (X - y)$. Then, whenever $\ell_{i,i+1}$ is non identically zero, there exists λ_i in $k - \{0\}$ such that, for all y in V , $\ell_{i,i+1}(y) = \lambda_i \mathcal{L}_{i,i+1}(y)$. In this case, $V = Z(\lambda_i^p \mathcal{L}_{i,i+1}^p - \lambda_i \mathcal{L}_{i,i+1})$.

Proof: The matricial multiplication first ensures that for all i in $\{1, \dots, n-1\}$, $\ell_{i,i+1}$ is a linear form from V to \mathbb{F}_p . Besides, from the preliminary remarks of section 1, we infer that $P_V(X) := \prod_{y \in V} (X - y)$ is a separable additive polynomial of degree p^v , where v denotes the dimension of the \mathbb{F}_p -vector space V . Then, for all i in $\{1, \dots, n-1\}$, $\mathcal{L}_{i,i+1}(X) := \prod_{y \in \text{Ker} \ell_{i,i+1}} (X - y)$ is an additive polynomial which divides $P_V(X)$. We now assume that $\ell_{i,i+1}$ is a nonzero linear form. In this case, $\mathcal{L}_{i,i+1}(X)$ has degree p^{v-1} and there exists λ_i in $k - \{0\}$ such that for all y in V , $\ell_{i,i+1}(y) = \lambda_i \mathcal{L}_{i,i+1}(y)$. Since for all y in V , $\ell_{i,i+1}(y)$ lies in \mathbb{F}_p , then $\lambda_i^p \mathcal{L}_{i,i+1}^p - \lambda_i \mathcal{L}_{i,i+1} = \lambda_i^p P_V$. The claim follows. \square

Remark 2.10. By duality with respect to the Artin-Schreier pairing, the adapted basis of A fixed in Definition 2.3.4 gives a basis of G_2 , say $\{g_1, \dots, g_n\}$, in which, the matrix of the automorphism $\phi(y)$ is the transpose matrix of $L(y)$ for all y in V , namely a lower triangular matrix of $GL_n(\mathbb{F}_p)$ with identity on the diagonal.

Proposition 2.11. Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. We keep the notation defined above. For all integer d such that $1 \leq d \leq n$, we denote by A_d the \mathbb{F}_p -subvector space of A generated by $\{f_i(X), 1 \leq i \leq d\}$. Let H_d be the orthogonal of A_d with respect to the Artin-Schreier pairing, namely the \mathbb{F}_p -subvector space of G_2 spanned by $\{g_i, d+1 \leq i \leq n\}$ if $d < n$ and $H_n = \{0\}$. Then, the pair $(C/H_d, G/H_d)$ is a big action such that $(\frac{G}{H_d})_2 = \frac{G_2}{H_d}$. It follows that $|\frac{G}{G_2}| = |\frac{G/H_d}{(G/H_d)_2}|$ and that the exact sequence

$$0 \longrightarrow G_2 \longrightarrow G \xrightarrow{\pi} V \longrightarrow 0$$

induces the following one:

$$0 \longrightarrow (G/H_d)_2 \simeq (\mathbb{Z}/p\mathbb{Z})^d \longrightarrow G/H_d \xrightarrow{\pi} V \longrightarrow 0$$

Proof: Since $\rho(V) \subset T_{1,n}^u(\mathbb{F}_p)$, A_d is stable under the action of ρ , that is to say under the translation: $X \rightarrow X + y$, with $y \in V$. By duality, H_d is stable under the action of ϕ , i.e. by conjugation by the elements of G_1 . It follows that H_d is a subgroup of G_2 , normal in G_1 . In this case, [MR08] (see Lemma 2.4 and Theorem 2.6) implies that the pair $(C/H_d, G/H_d)$ is a big action with $(\frac{G}{H_d})_2 = \frac{G_2}{H_d} \subset \frac{G}{H_d}$. The claim follows. \square

Corollary 2.12. Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. Let the functions $f_i(X) \in k[X]$ be as in section 2.3.1. Then, $f_1(X) = X S_1(X) + cX$, where $S_1 \in k\{F\}$ is an additive polynomial. Furthermore, after an homothety and a translation on X , one can assume that S_1 is monic and $c = 0$.

Proof: The function field of the curve C/H_d , as defined in Proposition 2.11, is parametrized by the d Artin-Schreier equations: $W_i^p - W_i = f_i(X)$, with $1 \leq i \leq d$. In particular, for $d = 1$ $(C/H_1, G/H_1)$ is a big action whose second lower ramification group has order p . Then, [LM05] asserts that $f_1(X) = X S_1(X) + cX$ in $k[X]$, where $S_1 \in k\{F\}$ is an additive polynomial. \square

2.5 Characterization of the trivial representation.

To conclude this section, we give a characterization of the case where the representation ρ or ϕ is trivial.

Proposition 2.13. *Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. When keeping the notation defined above, the following assertions are equivalent.*

1. *The representation ϕ is trivial, namely $\phi(V) = \{id\}$.*
2. *The second ramification group G_2 is included in the center of G_1 .*
3. *The representation ρ is trivial, i.e.*

$$\forall i \in \{1, \dots, n\}, \quad \forall y \in V, \quad f_i(X+y) - f_i(X) = 0 \pmod{\wp(k[X])}$$

4. *For all i in $\{1, \dots, n\}$, the functions f_i read: $f_i(X) = X S_i(X) + c_i X \pmod{\wp(k[X])}$, where S_i is an additive polynomial of degree $s_i \geq 1$ in F . Write $S_i(F) = \sum_{j=0}^{s_i} a_{i,j} F^j$ with $a_{i,s_i} \neq 0$. Then, following [El99] (section 4), we can define an additive polynomial related to f_i , called the "palindromic polynomial" of f :*

$$Ad_{f_i} := \frac{1}{a_{i,s_i}} F^{s_i} \left(\sum_{j=0}^{s_i} a_{i,j} F^j + F^{-j} a_{i,j} \right).$$

In this case,

$$V \subset \bigcap_{i=1}^n Z(Ad_{f_i})$$

The proof of this proposition requires a preliminary lemma.

Lemma 2.14. *When keeping the notation defined above, $\cap_{y \in V} \text{Ker}(\phi(y) - id) = Z(G_1) \cap G_2$.*

Proof of Lemma 2.14: Consider g in G_2 . Then, g lies in $\cap_{y \in V} \text{Ker}(\phi(y) - id)$ if and only if $\phi(y)(g) = g$ for all y in V . For all g_1 in G , put $y_1 := \pi(g_1)$. By definition, the equality $\phi(y_1)(g) = g$ means that $g_1^{-1} g g_1 = g$. This proves the expected formula. \square .

Proof of Proposition 2.13: The equivalence between the first and the second assertion derives from Lemma 2.14. As the equivalence between the first and the third point comes from the duality of ϕ and ρ (cf. Proposition 2.2), the only point that has to be explained is the equivalence between the last assertion and the three preceding ones.

For all i in $\{1, \dots, n\}$, the function field of the curve $W_i^p - W_i = f_i(X)$ is a p -cyclic étale cover of the affine line, whose Galois group is denoted by H_i . Then, H_i is an index p -subgroup of G_2 . Besides, if the second point is satisfied, G_2 is included in $Z(G_1)$, which implies that H_i is normal in G_1 . From [MR08] (see Lemma 2.4 and Theorem 2.6), we infer that $(C/H_i, G/H_i)$ is a big action whose second ramification group $(G/H_i)_2 = G_2/H_i$ is p -cyclic. By [LM05] (see Prop. 8.3), $f_i(X) = c_i X + X S_i(X) \pmod{\wp(k[X])}$, with $S_i \in k\{F\}$. In addition, V is included in $Z(Ad_{f_i})$. Conversely, if $f_i(X) = X S_i(X) + c_i X$, then it follows from Proposition 5.5 in [LM05] that $Z(Ad_{f_i}) = \{y \in k, f_i(X+y) - f_i(X) = 0 \pmod{\wp(k[X])}\}$. Thus, the third point is verified. \square

3 The link with the k -algebra generated by additive polynomials.

The purpose of this section is to highlight the role played by the k -algebra generated by the additive polynomials in the parametrization of big actions with a p -elementary abelian second ramification group.

3.1 The k -algebra generated by additive polynomials.

Definition 3.1. *We define Σ_1 as the k -subvector space of $k[X]$ generated by 1 and by the additive polynomials of $k[X]$. More generally, for any $n \geq 1$, we define Σ_n as the k -subvector space of $k[X]$ generated by 1 and the products of at most n additive polynomials of $k[X]$. For $n = 0$, we put $\Sigma_0 = k$ and for $n < 0$, we put $\Sigma_n = \{0\}$.*

Remark 3.2. 1. For $n \geq 1$, this definition means that f is a polynomial of Σ_n if and only if there is a way to write f as a linear combination over k of products of at most n additive polynomials.

2. The sequence $(\Sigma_n)_{n \in \mathbb{Z}}$ enjoys the following properties:

- (a) $1 \in \Sigma_0$
- (b) For all integer n in \mathbb{Z} , $\Sigma_n \subset \Sigma_{n+1}$
- (c) For all integer m and n in \mathbb{Z} , $\Sigma_m \Sigma_n \subset \Sigma_{m+n}$.
- (d) $\bigcup_{n \in \mathbb{Z}} \Sigma_n = k[X]$

In particular, the sequence $(\Sigma_n)_{n \in \mathbb{Z}}$ is an increasing ring filtration of $k[X]$.

For a given f in $k[X]$, we search for the minimal integer n such that f belongs to Σ_n . It requires the introduction of the order function related to the ring filtration.

Definition 3.3. Let a be an integer whose p -adic expansion reads: $a = a_0 + a_1 p + a_2 p^2 + \cdots + a_t p^t$, with $t \in \mathbb{N}$ and $0 \leq a_i \leq p-1$, for all $i \in \{0, 1, 2, \dots, t\}$. We define the integer $S_p(a) \in \mathbb{N}$ as the sum of the digits of a , namely:

$$S_p(a) := a_0 + a_1 + a_2 + \cdots + a_t.$$

Remark 3.4. For all integer m in \mathbb{N} , $S_p(m) = (p-1) v_p(m!)$, where v_p denotes the p -adic valuation. We gather that, if m_1 and m_2 are two non-negative integers, $S_p(m_1 + m_2) \leq S_p(m_1) + S_p(m_2)$.

Lemma 3.5. We keep the same notation as above. Let $a \in \mathbb{N}$ and $n \in \mathbb{N}$. Then, the monomial X^a lies in Σ_n if and only if $S_p(a) \leq n$. It follows that $\inf\{n \in \mathbb{N}, X^a \in \Sigma_n\} = S_p(a)$.

Proof: Assume that $X^a \in \Sigma_n$. It means that X^a is a linear combination over k of monomials of the form $X^{p^{\gamma_1} + p^{\gamma_2} + \cdots + p^{\gamma_t}}$, with $t \leq n$ and $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_t$. It follows that a also reads $a = p^{\alpha_1} + p^{\alpha_2} + \cdots + p^{\alpha_t}$ with $t \leq n$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_t$. Therefore, Remark 3.4 implies $S_p(a) = S_p(p^{\alpha_1} + p^{\alpha_2} + \cdots + p^{\alpha_t}) \leq S_p(p^{\alpha_1}) + S_p(p^{\alpha_2}) + \cdots + S_p(p^{\alpha_t}) = t \leq n$.

Conversely, we suppose that $S_p(a) \leq n$ and prove the result by induction on n . If $n = 0$, then $S_p(a) = 0$ and $X^a = X^0 = 1 \in \Sigma_0$. We now assume that the property is true for n and suppose that $S_p(a) \leq n+1$. If $S_p(a) = n$, then, by induction hypothesis, $X^a \in \Sigma_n \subset \Sigma_{n+1}$. Otherwise, $S_p(a) = n+1$ and there exists an integer a_i in the p -adic expansion of a such that $a_i \geq 1$. Put $b := a - p^i$. As $S_p(b) = n$, the hypothesis implies $X^b \in \Sigma_n$, hence $X^a = X^b X^{p^i} \in \Sigma_{n+1}$. \square

Definition 3.6. Let f be a nonnull polynomial of $k[X]$ such that $f(X) = \sum_{a \in \mathbb{N}} c_a(f) X^a$. We define

$$d_p(f) := \max_{c_a(f) \neq 0} \{S_p(a)\}$$

By convenience, put $d_p(0) := -\infty$.

Lemma 3.7. Let f and g be polynomials of $k[X]$. Let $n \in \mathbb{Z}$. We keep the same notation as above.

- 1. $f \in \Sigma_n$ if and only if $d_p(f) \leq n$. In other words, $f(X) = \sum_{a \in \mathbb{N}} c_a(f) X^a \in \Sigma_n$ if and only if, whenever $S_p(a) > n$, $c_a(f) = 0$.
- 2. If f is non identically zero, $d_p(f) = \inf\{n \in \mathbb{Z}, f \in \Sigma_n\}$.
- 3. $d_p(f) = -\infty$ if and only if $f \in \bigcap_{n \in \mathbb{Z}} \Sigma_n = \{0\}$.
- 4. $d_p(fg) \leq d_p(f) + d_p(g)$.
- 5. $d_p(f+g) \leq \sup\{d_p(f), d_p(g)\}$.
- 6. $d_p(F(f)) = d_p(f)$, where F means the Frobenius operator.
- 7. Let $S(X) \in k[X]$ be an additive polynomial. Then, $d_p(f(S(X))) = d_p(f(X))$.

In particular, d_p is the order function of the ring filtration defined by the $(\Sigma_n)_{n \in \mathbb{Z}}$.

Proof: Most of the properties can be deduced from Remark 3.4 and Lemma 3.5. The last one is left as an exercise to the reader. \square

Definition 3.8. Let f be a polynomial of $k[X]$. Let $y \in k$. We define the operator Δ_y as follows: $\Delta_y(f) := f(X + y) - f(X)$.

One checks that this operator enjoys the following property:

Lemma 3.9. For all y in k and for all $n \in \mathbb{Z}$, $\Delta_y(\Sigma_{n+1}) \subset \Sigma_n$.

Remark 3.10. Although $d_p(\Delta_y(X^a)) = d_p(X^a) - 1$, for all y in $k - \{0\}$ and all a in \mathbb{N}^* , one can find some polynomial f in $k[X]$ and some y in $k - \{0\}$ such that $d_p(\Delta_y(f)) \neq d_p(f) - 1$. It means that for $n \geq 2$ and for y in $k - \{0\}$, $\Delta_y(\Sigma_{n+1} - \Sigma_n)$ is not always included in $\Sigma_n - \Sigma_{n-1}$.

3.2 Notation and preliminary lemmas.

We begin by recalling some notation and proving some lemmas useful for the proof of next theorem. Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. We call condition (N) the inequality satisfied by big actions, namely: $\frac{|G|}{g} > \frac{2p}{p-1}$. We fix an adapted basis of A : $\{\overline{f_1(X)}, \dots, \overline{f_n(X)}\}$, as constructed in Definition 2.3 and assume that the functions $f_i(X)$ are reduced mod $\wp(k[X])$ (see definition in section 1). We denote by m_i the degree of $f_i(X)$. As recalled in Corollary 2.12, $f_1(X) = X S_1(X) + c_1 X$, where $S_1 \in k\{F\}$ is an additive polynomial with degree $s_1 \geq 1$ in F . In this case, the palindromic polynomial Ad_{f_1} related to f_1 is defined as in Proposition 2.13. Besides, the function field $L := k(C)$ is parametrized by the n Artin-Schreier equations: $W_i^p - W_i = f_i(X)$, with $1 \leq i \leq n$. We denote by ρ the representation from V to $\text{Aut}(A)$ defined in section 2.2. In the adapted basis fixed above, the automorphism $\rho(y)$ is associated with the unipotent matrix:

$$L(y) := \begin{pmatrix} 1 & \ell_{1,2}(y) & \ell_{1,3}(y) & \cdots & \ell_{1,n}(y) \\ 0 & 1 & \ell_{2,3}(y) & \cdots & \ell_{2,n}(y) \\ 0 & 0 & \cdots & \cdots & \ell_{i,n}(y) \\ 0 & 0 & \cdots & 1 & \ell_{n-1,n}(y) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in Gl_n(\mathbb{F}_p)$$

Lemma 3.11. We keep the notation defined above. The dimension of the \mathbb{F}_p -vector space V satisfies $v \leq 2s_1$ and $p^v \geq m_n + 1$. In particular, $2 \leq s_1 + 1 \leq v \leq 2s_1$.

Proof: The inclusion of V in $Z(Ad_{f_1})$ first requires $v \leq 2s_1$. On the one hand, $|G| = p^{n+v}$. On the other hand, Corollary 2.7 implies: $g = \frac{p-1}{2} \sum_{i=1}^n p^{i-1} (m_i - 1) \geq \frac{p-1}{2} p^{n-1} (m_n - 1)$. Thus, $\frac{|G|}{g} \leq \frac{2p}{p-1} \frac{p^v}{m_n-1}$. The inequality $p^v \leq m_n - 1$ would contradict condition (N). Therefore, since m_n is prime to p , we obtain $p^v \geq m_n + 1$. It follows that $p^v > m_n \geq m_1 = 1 + p^{s_1}$ and $v \geq s_1 + 1$. \square

Lemma 3.12. Let $f(X) := \sum_{a \in \mathbb{N}} c_a(f) X^a$ be a polynomial in $\wp(k[X])$. Fix $a_0 \in \mathbb{N} - p\mathbb{N}$ and define $I_{a_0} := \{a_0 p^n, n \in \mathbb{N}\}$. Then, the polynomial $f_{a_0}(X) := \sum_{a \in I_{a_0}} c_a(f) X^a$ also lies in $\wp(k[X])$. In particular, if $f_{a_0}(X)$ is non identically zero, then p divides its degree.

Proof: The Frobenius operator F acts on the basis $(X^a)_{a \in \mathbb{N}}$ of $k[X]$ and this action induces a partition of the monomials of $k[X]$, namely $(X^a)_{a \in I_{a_0}}$, for a_0 running over $\{0\} \cup \{\mathbb{N} - p\mathbb{N}\}$. This justifies the first claim. Now, assume that $f_{a_0}(X)$ is non identically zero. If $f = \wp(g)$ with $g \in k[X]$, then $f_{a_0} = \wp(g_{a_0})$, with g_{a_0} defined as for f . It follows that $\deg(f_{a_0}) = p \deg(g_{a_0})$. \square

3.3 The link with the parametrization of big actions.

Theorem 3.13. We keep the notation defined in sections 3.1 and 3.2. For all i in $\{1, \dots, n\}$, $f_i(X)$ belongs to Σ_{i+1} .

Proof: For a fixed n , we proceed by induction on i . As recalled above, $f_1(X) = X S_1(X) + c_1 X$, where S_1 is an additive polynomial. Accordingly, $f_1 \in \Sigma_2$. We now consider some integer i such that $2 \leq i \leq n$ and assume that for all j in $\{1, \dots, i-1\}$, $f_j(X)$ lies in Σ_{j+1} . From the form of the matrix $L(y)$, we gather:

$$\forall y \in V, \Delta_y(f_i) := f_i(X + y) - f_i(X) = \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X) \quad \text{mod } (\wp(k[X]))$$

where for all j in $\{1, \dots, i-1\}$, $\ell_{j,i}$ is a map from V to \mathbb{F}_p .

Suppose that $f_i(X)$ does not belong to Σ_{i+1} and call X^a the monomial of $f_i(X)$ with highest

degree which does not belong to Σ_{i+1} . Note that, by definition of a , $a \geq i + 1$. Furthermore, as f_i is assumed to be reduced mod $\wp(k[X])$, $a \not\equiv 0 \pmod p$.

We first prove that p divides $a - 1$. Indeed, assume that p does not divide $a - 1$ and apply Lemma 3.12 to $f(X) := \Delta_y(f_i) - \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ and $a_0 := a - 1 \in \mathbb{N} - p\mathbb{N}$. To construct the polynomial f_{a_0} as defined in Lemma 3.12, we first search for the monomials $X^{(a-1)p^r}$, with $r \geq 0$, in $\Delta_y(f_i)$. If $r > 0$, such monomials come from monomials X^b of $f_i(X)$ such that $b > (a - 1)p^r \geq (a - 1)p \geq a$, since $a \geq i + 1 \geq 2 \geq \frac{p}{p-1}$. By definition of a , such monomials X^b , whose degree is strictly higher than a , lies in Σ_{i+1} . Then, by Lemma 3.9, they generate in $\Delta_y(f_i)$ polynomials which belongs to Σ_i . But $X^{a-1} \notin \Sigma_i$: otherwise, $X^a \in X\Sigma_i \subset \Sigma_{i+1}$, which contradicts the definition of a . We infer from Lemma 3.7.6 that no $X^{(a-1)p^r}$, with $r \geq 0$, lies in Σ_i . It follows that no monomial $X^{(a-1)p^r}$, with $r > 0$, can be found in $\Delta_y(f_i)$. We now search for the monomial X^{a-1} . By the same token, one can check that the only monomial in $f_i(X)$ which generates X^{a-1} in $\Delta_y(f_i)$ is X^a . More precisely, it produces $ayc_a(f_i)X^{a-1}$ in $\Delta_y(f_i)$, where $c_a(f_i) \neq 0$ denotes the coefficient of X^a in f_i . As the induction hypothesis asserts that $\sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ lies in Σ_i , which is the case of none of the $X^{(a-1)p^r}$, we gather that $f_{a_0}(X) = ay c_a(f_i) X^{a-1}$. As p does not divide $a_0 = a - 1$, it follows from Lemma 3.12 that $f_{a_0}(X)$ is identically zero. Since $a \not\equiv 0 \pmod p$, this implies that $y = 0$ for all y in V , hence $V = \{0\}$. It means that $G_1 = G_2$, which is impossible for a big action. Accordingly, p divides $a - 1$. Thus, we can write $a = 1 + \lambda p^t$ with $t \geq 1$, λ prime to p and $\lambda > i \geq 2$, as X^a does not lie in Σ_{i+1} .

Now, put $j_0 := a - p^t = 1 + (\lambda - 1)p^t$ and apply Lemma 3.12 to $f(X) := \Delta_y(f_i) - \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ and $a_0 := j_0 \in \mathbb{N} - p\mathbb{N}$. To construct the polynomial f_{a_0} , we first determine the monomials $X^{j_0 p^r}$, with $r \geq 0$, occuring in $\Delta_y(f_i)$. If $r > 0$, such terms come from monomials X^b of $f_i(X)$ such that $b > j_0 p$. But $j_0 p > a$. Indeed,

$$j_0 p \leq a \Leftrightarrow p(1 + (\lambda - 1)p^t) \leq 1 + \lambda p^t \Leftrightarrow \lambda \leq \frac{1 - p + p^{t+1}}{p^t(p - 1)} = \frac{-1}{p^t} + \frac{p}{p - 1} < \frac{p}{p - 1} \leq 2$$

which contradicts $\lambda \geq 2$. As explained above, the monomials X^b of $f_i(X)$, with $b > a$, produce polynomials in $\Delta_y(f_i)$ which belongs to Σ_i , whereas X^{j_0} does not belong to Σ_i . Otherwise, $X^a = X^{p^t} X^{j_0}$ would belong to Σ_{i+1} , hence a contradiction. We gather from Lemma 3.7.6 that no $X^{j_0 p^r}$, with $r \geq 0$, lies in Σ_{i+1} . It follows that no monomials $X^{j_0 p^r}$, with $r > 0$, can be found in $\Delta_y(f_i)$. Likewise, for $r = 0$, the only monomials of $f_i(X)$ which generates X^{j_0} in $\Delta_y(f_i)$ are those of the form: X^b , with $j_0 + 1 \leq b \leq a$. For all $b \in \{j_0 + 1, \dots, a\}$, the monomial X^b of $f_i(X)$ generates some $\binom{b}{j_0} y^{b-j_0} X^{j_0}$ in $\Delta_y(f_i)$. It follows that the coefficient of X^{j_0} in $\Delta_y(f_i)$ is $T(y)$ with $T(Y) := \sum_{b=j_0+1}^a c_b(f_i) \binom{b}{j_0} Y^{b-j_0}$, where $c_b(f_i)$ denotes the coefficient of X^b in $f_i(X)$. As no $X^{j_0 p^r}$, with $r \geq 0$, can be found in $\sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ which lies in Σ_i by induction, the polynomial f_{a_0} eventually reads $f_{a_0}(X) = T(y) X^{a_0}$. By Lemma 3.12, f_{a_0} is identically zero, which means that for all y in V , $T(y) = 0$. We gather that V is included in the set of zeroes of T . As the coefficient of Y^{a-j_0} in $T(Y)$ is $c_a(f_i) \binom{a}{j_0} = c_a(f_i) \binom{1+\lambda p^t}{1+(\lambda-1)p^t} \equiv c_a(f_i) \lambda \not\equiv 0 \pmod p$, the polynomial $T(Y)$ has degree $a - j_0 = p^t$, hence $v \leq t$. This leads to a contradiction, insofar as Lemma 3.11 implies: $p^v \geq m_n - 1 \geq m_i - 1 \geq a - 1 = \lambda p^t \geq 2p^t > p^t$, which involves: $v > t$. As a consequence, $f_i(X)$ does not have any monomial which does not belong to Σ_{i+1} , which completes both the induction and the proof of the theorem. \square

Remark 3.14. *The proof is actually self-contained, since the first step of the induction, namely $f_1 \in \Sigma_2$, could be obtained without any hint at Corollary 2.12 which requires the use of [LM05]. Indeed, in the case $i = 1$, the sum $\sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ is replaced by 0 which obviously lies in Σ_1 . Using the same argument as in the second part of the proof, it enables us to conclude that f_1 belongs to Σ_2 .*

4 A special case of trivial representation.

This section is devoted to a first special case in which the representation ρ is trivial or, equivalently, each function f_i lies in $\Sigma_2 - \Sigma_1$. The difficulty in solving the general case of trivial representation lies in finding the GCD for the family of palindromic polynomials associated to the function f_i as defined in Proposition 2.13. This could be done by working in the Ore ring of Laurent polynomials $k\{F, F^{-1}\}$ (see [El99], section 3, or [Go96], 1.6). Nevertheless, in what follows, we merely explore the simplest case where all the palindromic polynomials are equal.

Lemma 4.1. Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. We keep the notation defined in sections 2.3.2 and 4.2. Then, the following assertions are equivalent.

1. The upper ramification filtration of G_2 has only one jump.
2. The functions f_i 's have the same degree, i.e. for all i in $\{1, \dots, n\}$, $m_i = m_1 = 1 + p^{s_1}$.

In this case, the representation ρ is trivial and each function f_i reads $f_i(X) = X S_i(X) + c_i X \in \Sigma_2 - \Sigma_1$, where S_i is an additive polynomial with degree s_1 in F . Moreover, $V \subset \cap_{1 \leq i \leq n} Z(Ad_{f_i})$.

Proof: Assume that there is only one jump in the upper ramification filtration of G_2 as defined in section 2.3.2, namely $G_2 = (G_2)^{\nu_0} \supsetneq (G_2)^{\nu_0+1} = \{0\}$. The duality between the filtrations of A and G_2 (cf. Proposition 2.5) implies that this is equivalent to $\{0\} = A^{\mu_0} \subsetneq A^{\mu_0+1} = A$. By Remark 2.4, this situation occurs if and only if all the functions $f_i(X)$ have the same degree, namely: $1 + p^{s_1}$. In this case, it follows from Remark 2.8 that the representation ρ is trivial. Then, the following assertions derive from Proposition 2.13. \square

In what follows, we restrict to the special case: $V = Z(Ad_{f_1})$, which means that V has maximal cardinality for a given s_1 , namely $|V| = p^{2s_1}$.

Proposition 4.2. Let $n \geq 2$. We assume that $\rho(V) = \{id\}$ and keep the notation defined above. We suppose that $v = 2s_n$. Then, for all i in $\{1, \dots, n\}$, $s_i = s$ and $V = Z(Ad_{f_1})$. Furthermore, there exists an integer d dividing s and some $\gamma_2, \dots, \gamma_n$ in $\mathbb{F}_{p^d} - \mathbb{F}_p$ such that:

$$S_1 = \sum_{j=0}^{s/d} a_{jd} F^{jd} \quad \text{and} \quad \forall i \in \{2, \dots, n\} \quad S_i = \gamma_i S_1$$

Moreover, $\{\gamma_1 := 1, \gamma_2, \dots, \gamma_n\}$ are linearly independent over \mathbb{F}_p . It follows that $s \geq 2$.

Proof: As $v \leq 2s_1 \leq 2s_n$, the hypothesis $v = 2s_n$ implies that each s_i is equal to s_1 . From now on, $s_1 = s_2 = \dots = s_n$ is denoted by s . By Proposition 2.13, $V \subset \cap_{1 \leq i \leq n} Z(Ad_{f_i}) \subset Z(Ad_{f_1})$. As the two vector spaces V and $Z(Ad_{f_1})$ have the same dimension over \mathbb{F}_p , namely $v = 2s$, we conclude that $Z(Ad_{f_1}) = V = Z(Ad_{f_i})$ for all i in $\{1, \dots, n\}$. Since k is algebraically closed and since Ad_{f_1} and Ad_{f_i} are monic, it follows that $Ad_{f_1} = Ad_{f_i}$.

Let i in $\{2, \dots, n\}$. Write: $S_1 = \sum_{k=0}^s a_k F^k$ and $S_i = \sum_{k=0}^s b_k F^k$, with $a_s \neq 0$ and $b_s \neq 0$. Then, $Ad_{f_1} = \frac{1}{a_s} F^s (\sum_{k=0}^s (a_k F^k + F^{-k} a_k)) = Ad_{f_i} = \frac{1}{b_s} F^s (\sum_{k=0}^s (b_k F^k + F^{-k} b_k))$. As for all $\alpha \in k$, $F \alpha = \alpha^p F$, we obtain: $\gamma_i \sum_{k=0}^s (a_k^{p^s} F^{s+k} + a_k^{p^{s-k}} F^{s-k}) = \sum_{k=0}^s (b_k^{p^s} F^{s+k} + b_k^{p^{s-k}} F^{s-k})$, with $\gamma_i a_k^{p^s} = b_k^{p^s}$ and $\gamma_i a_k^{p^{s-k}} = b_k^{p^{s-k}}$, for all $0 \leq k \leq s$. It implies that $\gamma_i^{p^k} = \gamma_i$, for all $0 \leq k \leq s$ such that $a_k \neq 0$. In particular, as $a_s \neq 0$, then $\gamma_i \in \mathbb{F}_{p^s}$. If we denote by d the degree of the minimal polynomial of γ_i over \mathbb{F}_p , then $\mathbb{F}_{p^d} := \mathbb{F}_p(\gamma_i) \subset \mathbb{F}_{p^s}$, so d divides s . By the same token, for all $0 \leq k \leq s$ such that $a_k \neq 0$, $\gamma_i \in \mathbb{F}_{p^k}$. Therefore, $\mathbb{F}_{p^d} = \mathbb{F}_p(\gamma_i) \subset \mathbb{F}_{p^k}$, which proves that d divides k , whenever $a_k \neq 0$. It follows that $S_1 = \sum_{j=0}^{s/d} a_{jd} F^{jd}$. In addition, as $\gamma_i \in \mathbb{F}_{p^s}$, we gather from $b_k^{p^s} = \gamma_i a_k^{p^s}$ for all $0 \leq k \leq s$, that $S_i = \gamma_i S_1$.

Note that $\{\gamma_1, \dots, \gamma_n\}$ are linearly independent over \mathbb{F}_p . Otherwise, it would contradict the condition (c) imposed in Definition 2.3.4. It follows that none of the γ_i 's, for $i \geq 2$, are in \mathbb{F}_p . So $s \geq 2$. \square

We now display a family of big actions satisfying the conditions described in Proposition 4.2.

Proposition 4.3. 1. Let $s \in \mathbb{N}^*$, $d \in \mathbb{N}^*$ dividing s and $n \in \mathbb{N}^*$ such that $n \leq d$.

Take $\{\gamma_1 := 1, \gamma_2, \dots, \gamma_n\}$ in \mathbb{F}_{p^d} , linearly independent over \mathbb{F}_p .

Put $S_1 := \sum_{j=0}^{s/d} a_{jd} F^{jd} \in k\{F\}$, with $a_s \neq 0$. For all i in $\{1, \dots, n\}$, we define $S_i := \gamma_i S_1$ in $k\{F\}$ and $f_i(X) := X S_i(X) + c_i X \in k[X]$. Then, for all i in $\{1, \dots, n\}$, $Z(Ad_{f_i}) = Z(Ad_{f_1})$. Put $V := Z(Ad_{f_1})$.

2. The function field of the curve C parametrized by: $W_i^p - W_i = f_i(X)$ with $1 \leq i \leq n$, is an étale extension of $k[X]$ with Galois group $H \simeq (\mathbb{Z}/p\mathbb{Z})^n$. Then, the group of translations of the affine line: $\{\tau_y : X \rightarrow X + y, y \in V\}$ extends to an automorphism p -group of C , say G , such that:

$$0 \longrightarrow H \simeq (\mathbb{Z}/p\mathbb{Z})^n \longrightarrow G \longrightarrow V \longrightarrow 0$$

3. Thus, we obtain a big action (C, G) whose second ramification group G_2 is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ and such that the representation ρ is trivial. Moreover, $Z(G) = D(G) = \text{Fratt}(G) \simeq (\mathbb{Z}/p\mathbb{Z})^n$, where $\text{Fratt}(G)$ denotes the Frattini subgroup of G and $D(G)$ its derived subgroup. Then, G is a special group (see [Su86], Def. 4.14). Besides, if $p > 2$, G has exponent p , whereas, if $p = 2$, G has exponent p^2 .

Proof:

1. Fix i in $\{1, \dots, n\}$. Then, $\gamma_i \text{Ad}_{f_1} = \frac{\gamma_i}{a_s} \sum_{j=0}^{s/d} (a_{jd} F^{jd} + F^{-jd} a_{jd})$. Since γ_i lies in \mathbb{F}_{p^d} , $\gamma_i \text{Ad}_{f_1} = \frac{1}{a_s} \sum_{j=0}^{s/d} (a_{jd} \gamma_i F^{jd} + F^{-jd} a_{jd} \gamma_i) = \frac{a_s \gamma_i}{a_s} \text{Ad}_{f_i} = \gamma_i \text{Ad}_{f_i}$. So, $Z(\text{Ad}_{f_1}) = Z(\text{Ad}_{f_i})$.
2. As $Z(\text{Ad}_{f_i}) = \{y \in k, \Delta_y(f_i) = 0 \pmod{\wp(k[X])}\}$ (see [LM05], Prop. 5.5), it follows that, for all y in V , $\Delta_y(f_i) = 0 \pmod{\wp(k[X])}$. So, Galois theory ensures the existence of the group G .
3. We deduce from the first point that $|G| = |G_2||V| = p^{n+2s}$. We compute the genus of C by means of the formula given in Corollary 2.7. This yields: $g = \frac{(p^n-1)p^s}{2}$. Therefore, $\frac{|G|}{g} = \frac{2p^{n+s}}{p^n-1}$. So, the pair (C, G) is a big action.

We now show that $Z(G) = D(G)$. By Proposition 2.13, $Z(G)$ contains G_2 which coincides with $D(G)$ (see Theorem 2.6 in [MR08]). Conversely, let \mathcal{H} be an index p -subgroup of $D(G)$. As $\mathcal{H} \subset D(G) \subset Z(G)$, \mathcal{H} is normal in G and Lemma 2.4.2 in [MR08] implies that the pair $(C/\mathcal{H}, G/\mathcal{H})$ is a big action. The curve C/\mathcal{H} is parametrized by an Artin-Schreier equation: $W^p - W = f(X) := \sum_{i=1}^n \lambda_i f_i(X)$, with $(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_p^n$ not all zeros. Condition (c) of Definition 2.3.4 imposes $\deg(f) = \max_{i=1, \dots, n} \{\deg \lambda_i f_i(X)\} = 1 + p^{s_1}$. Besides, by Proposition 2.11, we get the following exact sequence:

$$0 \longrightarrow D(G/\mathcal{H}) \simeq \mathbb{Z}/p\mathbb{Z} \longrightarrow G/\mathcal{H} \longrightarrow V \simeq (\mathbb{Z}/p\mathbb{Z})^{2s_1} \longrightarrow 0$$

In this case, Proposition 8.1 in [LM05] shows that G/\mathcal{H} is an extraspecial group, which involves $D(G)/\mathcal{H} = D(G/\mathcal{H}) = Z(G/\mathcal{H})$. We denote by $\pi : G \rightarrow G/\mathcal{H}$ the canonical mapping. Then, $\pi(Z(G)) \subset Z(G/\mathcal{H}) = D(G/\mathcal{H})$. As \mathcal{H} is included in $Z(G)$, it follows that $Z(G) \subset D(G)$. Since $Z(G) = D(G) = G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$ and since $D(G) = \text{Fratt}(G)$ for any p -group G , we gather that G is a special group. Moreover, if $p > 2$, Proposition 8.1 in [LM05] shows that G/\mathcal{H} is an extraspecial group with exponent p . Then, $\pi(G)^p = \{e\}$ and G^p is included in \mathcal{H} , for any hyperplanes \mathcal{H} of $D(G)$. It follows that $G^p = \{e\}$. If $p = 2$, the same proposition shows that G/\mathcal{H} has exponent p^2 . It implies that G also has exponent p^2 . \square

Remark 4.4. In the situation described in Proposition 4.3, $\frac{|G|}{g^2} = \frac{4p^n}{(p^n-1)^2}$. Note that the latter quotient does not depend on s any more.

Remark 4.5. For any big action (C, G) , the group G is included in the wild inertia subgroup of $\text{Aut}_k(C)$ at ∞ , denoted by $G_{\infty,1}$. Furthermore, Corollary 2.10 in [MR08] shows that the pair $(C, G_{\infty,1})$ is a big action with $D(G_{\infty,1}) = D(G)$. Now, assume that (C, G) is a big action as described in Proposition 4.3. Then, $p^{2s} = \frac{|G|}{|D(G)|} \leq \frac{|G_{\infty,1}|}{|D(G_{\infty,1})|} \leq p^{2s}$. It follows that, in this special case, G is equal to $G_{\infty,1}$.

5 The special case: f_i in $\Sigma_{i+1} - \Sigma_i$.

In this section, we define a filtration on the derived group $D(G)$ of any group G . Then, we investigate the case where G is extension of two elementary abelian p -groups and where the number of jumps in this filtration is maximal. Knowing that, for a big action, $G_2 = D(G)$ (see Theorem 2.6 in [MR08]), we apply these results to the case of big actions with a p -elementary abelian G_2 . This allows us to give a group-theoretic condition to characterize the big actions such that each function f_i lies in $\Sigma_{i+1} - \Sigma_i$. In this situation, we prove that the filtration on $D(G)$ actually coincides with the upper ramification filtration of G_2 as exposed in section 2.3.2. and that, as opposed to the previous case, the number of jumps in the filtration is maximal whereas the cardinality of V is minimal in regard to Lemma 3.11, namely: $v = s_1 + 1$.

5.1 A filtration on $D(G)$.

Definition 5.1. For any group G , we define a sequence of subgroups $(\Lambda_i(G))_{i \geq 0}$ as follows. Put $\Lambda_0(G) := \{e\}$, where e means the identity element of G . For all $i \geq 1$, let $\pi_{i-1} : G \rightarrow \frac{G}{\Lambda_{i-1}(G)}$ be

the canonical mapping. Then, $\Lambda_i(G)$ is the subgroup of G defined by $\pi_{i-1}^{-1}(Z(\frac{G}{\Lambda_{i-1}(G)}) \cap D(\frac{G}{\Lambda_{i-1}(G)}))$. Therefore,

$$\frac{\Lambda_i(G)}{\Lambda_{i-1}(G)} = Z(\frac{G}{\Lambda_{i-1}(G)}) \cap D(\frac{G}{\Lambda_{i-1}(G)}).$$

In this way, we get an ascending sequence of subgroups of $D(G)$:

$$\{e\} = \Lambda_0(G) \subset \Lambda_1(G) \subset \Lambda_2(G) \subset \cdots \subset D(G)$$

which are characteristic subgroups of G .

We study the filtration defined above in the special case where G is a p -group with the exact sequence:

$$0 \longrightarrow D(G) \simeq (\mathbb{Z}/p\mathbb{Z})^n \longrightarrow G \xrightarrow{\pi} V \simeq (\mathbb{Z}/p\mathbb{Z})^v \longrightarrow 0 \quad (2)$$

In other words, G is a p -group whose Frattini subgroup is equal to $D(G) \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. For convenience, we fix a set theoretical section, i.e. a map $s : G/D(G) \rightarrow G$ such that $\pi \circ s = id_{G/D(G)}$. We also define a representation $\phi : G/D(G) \rightarrow Aut(D(G))$ as follows. For all y in $G/D(G)$ and all g in $D(G)$, $\phi(y)(g) = s(y)^{-1} g s(y)$. As $G/D(G)$ is a p -group, one can find a basis $\{g_1, \dots, g_n\}$ of the \mathbb{F}_p -vector space $D(G)$ in which, for all y in $G/D(G)$, the matrix of the automorphism $\phi(y)$ belongs to the subgroup of $Gl_n(\mathbb{F}_p)$ made of the lower triangular matrices with identity on the diagonal, namely:

$$\Phi(y) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{2,1}(y) & 1 & 0 & \cdots & 0 \\ \ell_{3,1}(y) & \ell_{3,2}(y) & \cdots & 0 & 0 \\ \ell_{i,1}(y) & \ell_{i,2}(y) & \cdots & 1 & 0 \\ \ell_{n,1}(y) & \ell_{n,2}(y) & \cdots & \ell_{n,n-1}(y) & 1 \end{pmatrix} \in Gl_n(\mathbb{F}_p)$$

Note that for $n \geq 2$ and for all i in $\{1, \dots, n-1\}$, $\ell_{i+1,i}$ is a linear form from $G/D(G)$ to \mathbb{F}_p .

Proposition 5.2. *Let G be a group satisfying (2). We keep the notation defined above. Then, the following assertions are equivalent.*

1. The filtration defined by the $(\Lambda_i)_{i \geq 0}$ satisfies:

$$\{e\} = \Lambda_0(G) \subsetneq \Lambda_1(G) \subsetneq \Lambda_2(G) \subsetneq \cdots \subsetneq \Lambda_n = D(G)$$

which means, for all i in $\{1, \dots, n\}$,

$$\frac{\Lambda_i(G)}{\Lambda_{i-1}(G)} = Z(\frac{G}{\Lambda_{i-1}(G)}) \cap D(\frac{G}{\Lambda_{i-1}(G)}) \simeq \mathbb{Z}/p\mathbb{Z}.$$

2. For all i in $\{1, \dots, n\}$, $\Lambda_i(G)$ is the \mathbb{F}_p -subvector space of $D(G)$ spanned by $\{g_{n-i+1}, \dots, g_n\}$.
3. For $n \geq 2$ and for all i in $\{1, \dots, n-1\}$, $\ell_{i+1,i}$ is a nonzero linear form.

Proof: We prove that the first point implies the second one by induction on i . Assume $i = 1$. By the same argument as in Lemma 2.14, one proves that $\Lambda_1(G) = Z(G) \cap D(G)$ is equal to $\cap_{y \in G/D(G)} Ker(\phi(y) - id)$. Then, the form of $\Phi(y)$ shows that $\cap_{y \in G/D(G)} Ker(\phi(y) - id)$ contains the \mathbb{F}_p -vector space spanned by g_n . As $\Lambda_1(G)$ is assumed to be isomorphic to $\mathbb{Z}/p\mathbb{Z}$, it follows that $\Lambda_1(G) = \mathbb{F}_p g_n$. Now, take $i \geq 2$ and assume that $\Lambda_{i-1}(G)$ is the \mathbb{F}_p -subvector space of $D(G)$ spanned by $\{g_{n-i+2}, \dots, g_n\}$. Then, $\frac{G}{\Lambda_{i-1}(G)}$ is a p -group with the following exact sequence:

$$0 \longrightarrow \frac{D(G)}{\Lambda_{i-1}(G)} = D(\frac{G}{\Lambda_{i-1}(G)}) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-i+1} \longrightarrow \frac{G}{\Lambda_{i-1}(G)} \xrightarrow{\pi} G/D(G) \simeq (\mathbb{Z}/p\mathbb{Z})^v \longrightarrow 0$$

This exact sequence induces a representation $\phi_{i-1} : G/D(G) \rightarrow Aut(\frac{D(G)}{\Lambda_{i-1}(G)})$. Consider the canonical mapping: $\pi_{i-1} : D(G) \rightarrow \frac{D(G)}{\Lambda_{i-1}(G)}$. In the basis $\{\pi_{i-1}(g_1), \dots, \pi_{i-1}(g_{n-i+1})\}$, the matrix $\Phi_{i-1}(y)$ of the automorphism $\phi_{i-1}(y)$ reads:

$$\Phi_{i-1}(y) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{2,1}(y) & 1 & 0 & \cdots & 0 \\ \ell_{3,1}(y) & \ell_{3,2}(y) & \cdots & 0 & 0 \\ \ell_{i,1}(y) & \ell_{i,2}(y) & \cdots & 1 & 0 \\ \ell_{n-i+1,1}(y) & \ell_{n-i+1,2}(y) & \cdots & \ell_{n-i+1,n-i}(y) & 1 \end{pmatrix} \in Gl_{n-i+1}(\mathbb{F}_p)$$

where the maps $\ell_{i,j}$ are the same as in $\Phi(y)$. As in the case $i = 1$, $\frac{\Lambda_i(G)}{\Lambda_{i-1}(G)} = Z(\frac{G}{\Lambda_{i-1}(G)}) \cap D(\frac{G}{\Lambda_{i-1}(G)})$ is equal to $\cap_{y \in G/D(G)} \text{Ker}(\phi_{i-1}(y) - id)$. The latter is the \mathbb{F}_p -vector space of $D(\frac{G}{\Lambda_{i-1}(G)}) = \frac{D(G)}{\Lambda_{i-1}(G)}$ generated by $\pi_{i-1}(g_{n-i+1})$. It follows that $\Lambda_i(G)$ is the \mathbb{F}_p -subvector space of $D(G)$ spanned by $\{g_{n-i+1}, \dots, g_n\}$. As the second assertion trivially implies the first one, the equivalence between 1 and 2 is established.

We now prove that the second assertion implies the third one. Take $i \geq 1$. As seen above, $\frac{\Lambda_i(G)}{\Lambda_{i-1}(G)} = \cap_{y \in G/D(G)} \text{Ker}(\phi_{i-1}(y) - id)$ is the \mathbb{F}_p -vector space spanned by $\pi_{i-1}(g_{n-i+1})$. From the form of the matrix $\Phi_{i-1}(y)$, we gather that $\ell_{n-i+1, n-i}$ is non identically zero. The proof of the converse works by induction on i . If $i = 1$, the form of the matrix $\Phi(y)$, with each $\ell_{i+1, i}$ non identically zero, implies that $\Lambda_1(G) = \cap_{y \in G/D(G)} \text{Ker}(\phi(y) - id)$ is the \mathbb{F}_p -subvector space of $D(G)$ spanned by g_n . Now, take $i \geq 2$ and assume that $\Lambda_{i-1}(G)$ is the \mathbb{F}_p -subvector space of $D(G)$ spanned by $\{g_{n-i+2}, \dots, g_n\}$. By hypothesis, each linear form $\ell_{i+1, i}$ occuring in $\Phi_{i-1}(y)$ is non identically zero. It implies that $\frac{\Lambda_i(G)}{\Lambda_{i-1}(G)} = \cap_{y \in G/D(G)} \text{Ker}(\phi_{i-1}(y) - id)$ is the \mathbb{F}_p -vector space spanned by $\pi_{i-1}(g_{n-i+1})$. We conclude as above. \square

Remark 5.3. Note that the third condition of the Proposition 5.2 does not actually depend on the triangularization basis $\{g_1, \dots, g_n\}$ chosen for $D(G)$.

Proposition 5.4. Let G be a group satisfying (2) and the equivalent properties of Proposition 5.2. We assume that $n \geq 2$.

1. For all i in $\{2, \dots, n\}$, there exists $\lambda_i \in \mathbb{F}_p - \{0\}$ such that $\ell_{i+1, i} = \lambda_i \ell_{2, 1}$. Therefore, one can choose a basis of $D(G)$ in which the matrix $\Phi(y)$ reads as follows:

$$\Phi(y) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell(y) & 1 & 0 & \cdots & 0 \\ \ell_{3,1}(y) & \ell(y) & \cdots & 0 & 0 \\ \ell_{i,1}(y) & \ell_{i,2}(y) & \cdots & 1 & 0 \\ \ell_{n,1}(y) & \ell_{n,2}(y) & \cdots & \ell(y) & 1 \end{pmatrix}$$

where ℓ is a nonzero linear form from $G/D(G)$ to \mathbb{F}_p .

2. Furthermore, $n \leq p$.

Proof: As $G/D(G)$ is abelian, for all y and y' in V , $\Phi(y)\Phi(y') = \Phi(y')\Phi(y)$. Then, for all i in $\{1, \dots, n-2\}$, the identification of the coefficients situated on the $(i+2)$ -th line and the i -th column in the matrices $\Phi(y)\Phi(y')$ and $\Phi(y')\Phi(y)$ reads:

$$\ell_{i+2, i}(y) + \ell_{i+1, i}(y') \ell_{i+2, i+1}(y) + \ell_{i+2, i}(y') = \ell_{i+2, i}(y') + \ell_{i+1, i}(y) \ell_{i+2, i+1}(y') + \ell_{i+2, i}(y)$$

Therefore, for all y and y' in $G/D(G)$, $\ell_{i+1, i}(y') \ell_{i+2, i+1}(y) = \ell_{i+1, i}(y') \ell_{i+2, i+1}(y)$. As $\ell_{i+1, i}$ and $\ell_{i+2, i+1}$ are nonzero linear forms, it follows that $\text{Ker} \ell_{i+1, i} = \text{Ker} \ell_{i+2, i}$. Then, $\ell_{i+1, i}$ and $\ell_{i+2, i+1}$ are homothetic. It implies that, for all i in $\{2, \dots, n\}$, there exists $\lambda_i \in \mathbb{F}_p - \{0\}$ such that $\ell_{i+1, i} = \lambda_i \ell_{2, 1}$. We eventually replace the basis of $D(G)$: $(g_i)_{1 \leq i \leq n}$ with $(\frac{1}{\lambda_i} g_i)_{1 \leq i \leq n}$ and denote $\ell_{2, 1}$ by ℓ . In this new basis, the matrix $\Phi(y)$ reads as expected and the first point is proved.

We now work with a basis of $D(G)$ in which the matrix $\Phi(y)$ reads as in the first point. We take some y_0 in $G/D(G)$ such that $\ell(y_0) \neq 0$. One checks that n is the smallest integer $m \geq 1$ such that $(\Phi(y_0) - I_n)^m = 0$, where I_n denotes the identity matrix of size n . But, as $G/D(G)$ has exponent p , then $(\Phi(y_0) - I_n)^p = \Phi(y_0)^p - I_n = 0$. It follows that $p \geq n$. \square

Remark 5.5. In the situation exposed in Proposition 5.2, that is to say in the case where each linear form $\ell_{i+1, i}$ in $\Phi(y)$ is non identically zero, the representation ϕ is said to be indecomposable, i.e. if $D(G) = D(G)_1 \oplus D(G)_2$, where $D(G)_1$ and $D(G)_2$ are two \mathbb{F}_p -subvectors spaces of $D(G)$ stable by ϕ , then the $D(G)_i$'s are trivial (left as an exercise to the reader). Nevertheless, the converse is false, i.e. the representation ϕ can be indecomposable without the linear forms $\ell_{i+1, i}$'s being all nonzero.

5.2 A group-theoretic characterization for big actions with $f_i \in \Sigma_{i+1} - \Sigma_i$.

In the sequel, we study the filtration defined by the $(\Lambda_i(G))_{i \geq 0}$ in the special case of a big action (C, G) whose G_2 is p -elementary abelian. Since G_2 coincides with $D(G)$ (see [MR08], Theorem 2.6), note that such a group G systematically satisfies condition (2). We now investigate the case where the group G satisfies the equivalent properties of Proposition 5.2. In particular, we show that these

group-theoretic conditions characterize the big actions with a p -elementary abelian G_2 and such that each f_i lies in $\Sigma_{i+1} - \Sigma_i$. The final section will be devoted to explicit families of big actions satisfying these properties. Throughout this section, the notations concerning big actions are those fixed in section 3.2.

Theorem 5.6. *Let (C, G) be a big action with $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, for $n \geq 2$, and such that the group G satisfies the equivalent properties of Proposition 5.2. Then, $p \geq n + 1 \geq 3$. Furthermore, for all i in $\{1, \dots, n\}$, $m_i = 1 + i p^{s_1}$. In particular, $f_i \in \Sigma_{i+1} - \Sigma_i$. Moreover, $v = s_1 + 1$. In this case,*

$$\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^n(p-1)^2}{n p^n(p-1) + 1 - p^n} > \frac{2p}{p-1}$$

Proof: For a fixed n , we prove by induction on i that for all i in $\{1, \dots, n\}$ such that $i \leq p-1$, $m_i = 1 + i p^{s_1}$. By the way, we show that $n \leq p-1$. Indeed, we cannot propagate the induction when $i = p-1$ and $n = p$.

The first step of the induction derives from the definition of m_1 . Then, we consider some integer i such that $2 \leq i \leq n$ and $i \leq p-2$ and assume that the proposition is true for all $j \leq i-1$. As seen in section 2.4, we can write:

$$\forall y \in V, \quad \Delta_y(f_i) := f_i(X+y) - f_i(X) = \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X) \quad \text{mod } \wp(k[X]) \quad (3)$$

where the maps $\ell_{j,i}$ from V to \mathbb{F}_p refer to the coefficients of the matrix $L(y)$. As the group G satisfies the third condition of Proposition 5.2 which does not depend on the basis chosen for $D(G)$, it follows from Proposition 2.9 and Remark 2.10 that for all i in $\{1, \dots, n-1\}$, each $\ell_{i,i+1}$ is a nonzero linear form from V to \mathbb{F}_p .

1. *We first prove that the function f_i does not belong to Σ_i .*

Assume that f_i lies in Σ_i and apply Lemma 3.12 to $f(X) := \Delta_y(f_i) - \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ and $a_0 := m_{i-1}$. By induction hypothesis, $m_{i-1} = 1 + (i-1)p^{s_1} \in \mathbb{N} - p\mathbb{N}$. Note that $X^{a_0} = X^{1+(i-1)p^{s_1}}$ lies in $\Sigma_i - \Sigma_{i-1}$. We gather from Lemma 3.7.6 that no $X^{a_0 p^r}$, with $r \geq 0$, belongs to Σ_{i-1} , so none of them can be found in $\Delta_y(f_i)$ which belongs to Σ_{i-1} , as f_i lies in Σ_i (cf. Lemma 3.9). Besides, the property c imposed on m_i by Definition 2.3.4 implies that, for any y in V such that $\ell_{i-1,i}(y) \neq 0$, $a_0 = m_{i-1}$ is the degree of $\sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$. Such an element y exists since $\ell_{i-1,i}$ is supposed to be a nonzero linear form. It follows that, when keeping the notation of Lemma 3.12, $f_{a_0}(X) = c_{m_{i-1}}(f_{i-1}) \ell_{i-1,i}(y) X^{a_0}$, where $c_{m_{i-1}}(f_{i-1}) \neq 0$ denotes the coefficient of $X^{m_{i-1}}$ in f_{i-1} . As p does not divide a_0 , we gather from Lemma 3.12 that $f_{a_0}(X)$ is identically zero, which contradicts $\ell_{i-1,i}(y) \neq 0$. Therefore f_i does not belong to Σ_i . In particular, as $\Sigma_2 \subset \Sigma_i$, f_i does not belong to Σ_2 . Accordingly, we can define an integer $a \leq m_i$ such that X^a is the monomial of f_i with highest degree which does not lie in Σ_2 . Since f_i is assumed to be reduced mod $\wp(k[X])$, $a \not\equiv 0 \pmod p$.

2. *We now prove that $a-1 \geq 1 + (i-1)p^{s_1}$.*

Assume that $a-1 < 1 + (i-1)p^{s_1}$ and apply Lemma 3.12 to $f(X) := \Delta_y(f_i) - \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ and $a_0 := m_{i-1} = 1 + (i-1)p^{s_1} \in \mathbb{N} - p\mathbb{N}$. The proof works as above except that we now have to determine the monomials of f_i which could produce some p -powers of X^{a_0} in $\Delta_y(f_i)$. As $a-1 < a_0$, they must be searched for among the monomials of f_i with degree strictly greater than a . But, by definition of a , such monomials belongs to Σ_2 , so give monomials in $\Delta_y(f_i)$ which are in Σ_1 , whereas $X^{a_0} = X^{1+(i-1)p^{s_1}}$ lies in $\Sigma_i - \Sigma_{i-1}$, with $i \geq 2$. Just as in the first point, we can conclude that, for any y in V such that $\ell_{i-1,i}(y) \neq 0$, $f_{a_0}(X) = c_{m_{i-1}}(f_{i-1}) \ell_{i-1,i}(y) X^{a_0}$, which leads to the same contradiction as above.

3. *We show that p divides $a-1$.*

Assume that p does not divide $a-1$. We first suppose that $a-1 > 1 + (i-1)p^{s_1}$ and apply Lemma 3.12 to $f(X) := \Delta_y(f_i) - \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ and $a_0 := a-1 \in \mathbb{N} - p\mathbb{N}$. As explained above, the monomials in f_i with degree stricly greater than a , produce in $\Delta_y(f_i)$ monomials which are in Σ_1 . But, as p does not divide $a-1$, the monomial X^{a-1} cannot belong to Σ_1 : otherwise, $a-1 = 1$, which contradicts $a-1 > 1 + (i-1)p^{s_1}$, with $i \geq 2$. So the only p -power of X^{a-1} that occur in $\Delta_y(f_i)$ comes from the monomial X^a of f_i : it is $c_a(f_i) a y X^{a-1}$, where $c_a(f_i) \neq 0$ denotes the coefficient of X^a in f_i . Besides, X^{a-1} does not occur in $\sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ whose degree is at most $1 + (i-1)p^{s_1} < a-1$. We gather from

Lemma 3.12 that $f_{a_0}(X) = c_a(f_i) a y X^{a_0}$ is identically zero. It implies that $V = \{0\}$, which is excluded for a big action. Accordingly, $a - 1 = 1 + (i - 1)p^{s_1}$. The equality of the leading coefficients in (3) implies that for all y in V , $\ell_{i-1,i}(y) = \frac{a c_a(f_i)}{c_{m_{i-1}}(f_{i-1})} y$. So the kernel of the linear form $\ell_{i-1,i}$ is reduced to $\{0\}$ and $v \leq 1$, which contradicts Lemma 3.11. Accordingly, p divides $a - 1$. Thus, we can write $a := 1 + \lambda p^t$, with $t > 0$, λ prime to p and $\lambda \geq 2$ because of the definition of a .

4. Put $j_0 := a - p^t = 1 + (\lambda - 1)p^t$. We prove that $j_0 = 1 + (i - 1)p^{s_1}$.
Indeed, if $j_0 < 1 + (i - 1)p^{s_1}$, then $a = j_0 + p^t < 1 + (i - 1)p^{s_1} + p^t$. Using the second point, we get: $1 + (i - 1)p^{s_1} < a = 1 + \lambda p^t < 1 + p^t + (i - 1)p^{s_1}$. If $s_1 - t \geq 0$, it implies $(i - 1)p^{s_1-t} < \lambda < 1 + (i - 1)p^{s_1-t}$ with $p^{s_1-t} \in \mathbb{N}$, which is impossible. So, $s_1 - t \leq -1$. In this case, as $i - 1 < p$, the inequality $1 + \lambda p^t < 1 + (i - 1)p^{s_1} + p^t$ yields: $\lambda - 1 < (i - 1)p^{s_1-t} < p^{1+s_1-t} \leq 1$, which contradicts $\lambda \geq 2$. As a consequence, $j_0 \geq 1 + (i - 1)p^{s_1}$.
We now prove that $j_0 = 1 + (i - 1)p^{s_1}$. Assume that $j_0 > 1 + (i - 1)p^{s_1}$ and apply Lemma 3.12 to $f(X) := \Delta_y(f_i) - \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ and $a_0 := j_0 \in \mathbb{N} - p\mathbb{N}$. No p -power of X^{j_0} can be found in $\sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X)$ whose degree is at most $1 + (i - 1)p^{s_1} < j_0$. It follows that the monomials $X^{j_0 p^r}$ have to be searched for in $\Delta_y(f_i)$. Then, the same argument as in the proof of Theorem 3.13 allows to write: $f_{a_0}(X) = T(y) X^{j_0}$ with $T(y) := \sum_{b=j_0+1}^a c_b(f_i) \binom{b}{j_0} y^{b-j_0}$, where $c_b(f_i)$ denotes the coefficient of X^b in $f_i(X)$. This entails the same contradiction with Lemma 3.11 as in the proof of Theorem 3.13. Therefore, $j_0 = 1 + (i - 1)p^{s_1}$.
5. We gather that $v = t + 1$.
Indeed, since $j_0 = 1 + (i - 1)p^{s_1} = \deg f_{i-1}$, the equality of the corresponding coefficients in (3) reads: $T(y) = \ell_{i-1,i}(y) c_{m_{i-1}}(f_{i-1})$, which holds for all y in V . Put $\tilde{T} := \frac{T}{c_{m_{i-1}}(f_{i-1})}$. It has the same degree as T and satisfies $\tilde{T}(y) = \ell_{i-1,i}(y) \in \mathbb{F}_p$, for all y in V . It follows that $\tilde{T}^p - \tilde{T}$ is identically zero on V , so $v \leq t + 1$. Using the same argument as in the proof of Theorem 3.13, we prove that $v \leq t$ contradicts Lemma 3.11. We gather that $v = t + 1$.
6. We prove that $s_1 = t$. It follows that $v = s_1 + 1$ and $a = 1 + i p^{s_1}$, which requires $p > n \geq 2$.
As $j_0 = 1 + (i - 1)p^{s_1}$, then $a = j_0 + p^t = 1 + (i - 1)p^{s_1} + p^t$. But, $a = 1 + \lambda p^t \geq 1 + 2p^t$. From $i - 1 \leq p$, we gather that $p^t \leq (i - 1)p^{s_1} < p^{s_1+1}$. Therefore, $t \leq s_1$. To prove the equality, we focus on the big action $(C/H_i, G/H_i)$ as defined in Proposition 2.11. Since $v = t + 1$, then $|G/H_i| = p^{i+v} = p^{i+t+1}$. Besides, as $m_i \geq a = j_0 + p^t = 1 + (i - 1)p^{s_1} + p^t \geq 1 + p^{s_1} + p^t$,

$$g_{C/H_i} \geq \frac{(p-1)}{2} p^{i-1} (m_i - 1) \geq \frac{(p-1)}{2} (p^{i-1+s_1} + p^{i-1+t})$$

If $u := s_1 - t \geq 1$, the lower bound for the genus becomes:

$$g_{C/H_i} \geq \frac{(p-1)}{2} p^{t+i-1} (p^u + 1) \geq \frac{(p-1)}{2} p^{t+i-1} (p + 1)$$

This contradicts condition (N) insofar as:

$$\frac{|G/H_i|}{g_{C/H_i}} \leq \frac{2p}{p-1} \frac{p^{i+t}}{p^{i-1+t}(p+1)} = \frac{2p}{p-1} \frac{p}{p+1} < \frac{2p}{p-1}$$

Therefore, $s_1 = t$, so $v = s_1 + 1$ and $a = 1 + (i - 1)p^{s_1} + p^t = 1 + i p^{s_1}$. Note that we find: $\lambda = i$. As λ is supposed to be prime to p and as $2 \leq i \leq n$ and $i \leq p - 1$, it requires that $p > n \geq 2$.

7. We conclude that $m_i = a = 1 + i p^{s_1}$.
Assume $a < m_i$. Then, by definition of a , there exists an integer $r \geq 0$ such that $m_i = 1 + p^r$. Thus, we get: $m_i = 1 + p^r > a = 1 + i p^{s_1} \geq 1 + 2p^{s_1}$. As $p \geq 3$, this implies $r \geq s_1 + 1$. We gather a new lower bound for the genus of C/H_i , namely:

$$g_{C/H_i} \geq \frac{(p-1)}{2} (p^{s_1} + p^{i-1} (m_i - 1)) = \frac{(p-1)}{2} (p^{s_1} + p^{i-1+r}) \geq \frac{(p-1)}{2} (1 + p^{i+s_1})$$

As $|G/H_i| = p^{i+s_1+1}$, it follows that $\frac{|G/H_i|}{g_{C/H_i}} \leq \frac{2p}{(p-1)} \frac{p^{i+s_1}}{1+p^{i+s_1}} < \frac{2p}{(p-1)}$, which contradicts condition (N) for the big action $(C/H_i, G/H_i)$. Accordingly, $m_i = a = 1 + i p^{s_1}$, which completes the induction.

To conclude, we compute the genus by means of Corollary 2.7, namely $g = \frac{p-1}{2} p^{s_1} \frac{p^n(p-1)+1-p^n}{(p-1)^2}$. It follows that

$$\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^n(p-1)^2}{n p^n(p-1)+1-p^n} \geq \frac{2p}{p-1} \frac{p^n(p-1)^2}{p^n(p-1)^2+1-p^n} > \frac{2p}{p-1} \quad \square$$

Corollary 5.7. *Let (C, G) be a big action as in Theorem 5.6. Let $G_{\infty,1}$ be the wild inertia subgroup of $\text{Aut}_k(C)$ at ∞ . Then, G is equal to $G_{\infty,1}$.*

Proof: As before, we denote by L the function field of C and by $k(X)$ the subfield of L fixed by $D(G)$. By [MR08] (Corollary 2.10), the pair $(C, G_{\infty,1})$ is a big action such that $D(G_{\infty,1}) = D(G)$. It follows that $G/D(G)$ is included in $G_{\infty,1}/D(G_{\infty,1})$, both of them acting as a group of translations of $\text{Spec } k[X]$. In the same way as we define the representation $\phi : G/D(G) \rightarrow \text{Aut}(D(G))$ in section 2.2 (or more generally in section 6.1), consider a representation $\tilde{\phi}$ from $G_{\infty,1}/D(G_{\infty,1})$ to $\text{Aut}(G_{\infty,1})$. Fix an adapted basis of A and then, by duality, a basis of $D(G)$ in which, for all y in $G/D(G)$ the matrix $\Phi(y)$ of the automorphism $\phi(y)$ is lower triangular. For all y in $G_{\infty,1}/D(G_{\infty,1})$, call $\tilde{\Phi}(y)$ the matrix of the automorphism $\tilde{\phi}(y)$ in the same adapted basis. When restricted to $G/D(G)$, the two matrices coincides, i.e. if y lies in $G/D(G) \subset G_{\infty,1}/D(G_{\infty,1})$, $\Phi(y) = \tilde{\Phi}(y)$. As a consequence, the group $G_{\infty,1}$ also satisfies the third condition of Proposition 5.2. Therefore, by Theorem 5.6, $\frac{|G_{\infty,1}|}{|D(G_{\infty,1})|} = s_1 + 1 = \frac{|G|}{|D(G)|}$. So, $G = G_{\infty,1}$. \square

We conclude this section by showing that the big actions (C, G) such that G satisfies the equivalent properties of Proposition 5.2 are exactly those with $f_i \in \Sigma_{i+1} - \Sigma_i$.

Theorem 5.8. *Let (C, G) be a big action with $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, for $n \geq 2$. We keep the notation defined in sections 4.1 and 4.2. Then, the following assertions are equivalent.*

1. *For all i in $\{1, \dots, n\}$, the function f_i lies in $\Sigma_{i+1} - \Sigma_i$.*
2. *The group G satisfies the equivalent properties of Proposition 5.2.*

In this case, $n \leq p-1$, $m_i = 1 + i p^{s_1}$ for all i in $\{1, \dots, n\}$, $v = s_1 + 1$ and $\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^n(p-1)^2}{n p^n(p-1)+1-p^n}$. Moreover, the upper ramification groups of G_2 coincide with the subgroups $\Lambda_i(G)$ studied in section 5.1. More precisely, following the notation of section 2.3.2, $(G_2)^{\nu_i} = \Lambda_{n-i}(G)$ for all i in $\{0, \dots, n\}$.

Proof: The implication from 2 to 1 comes from Theorem 5.6 which also shows that, in this case, $n \leq p-1$, $m_i = 1 + i p^{s_1}$, for all i in $\{1, \dots, n\}$, $v = s_1 + 1$ and $\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^n(p-1)^2}{n p^n(p-1)+1-p^n} > \frac{2p}{p-1}$. Conversely, assume that the second assertion is satisfied. We prove by induction on i that, for all i in $\{1, \dots, n-1\}$, the linear form $\ell_{i,i+1}$ is nonzero. Then, by Proposition 2.9, Remark 2.10 and Remark 5.3, we gather that the group G satisfies the third condition of Proposition 5.2. We first study the case $i = 1$ and consider the big action $(C/H_2, G/H_2)$, as defined in Proposition 2.11, i.e. the big action whose curve C/H_2 is parametrized by $W_j^p - W_j = f_j(X)$, with $1 \leq j \leq 2$. By hypothesis, f_2 does not lie in Σ_2 . We infer from Proposition 2.13 that the representation ρ associated with $(C/H_2, G/H_2)$ is non trivial. Then, the linear form $\ell_{1,2}$ is nonzero. We now take $i \geq 2$ and assume that the property is true for all $j \leq i$. It means that, for all i in $\{1, \dots, i-1\}$, the linear form $\ell_{j,j+1}$ is nonzero. Then, by Theorem 5.6, for all j in $\{1, \dots, i-1\}$, $m_j = 1 + j p^{s_1}$ and $v = s_1 + 1$. We now write condition (N) for the big action $(C/H_{i+1}, G/H_{i+1})$ as defined in Proposition 2.11, that is to say the big action parametrized by $W_j^p - W_j = f_j(X)$, with $1 \leq j \leq i+1$. As $|G/H_{i+1}| = p^{v+i+1} = p^{s_1+i+2}$ and $g_{C/H_{i+1}} = \frac{p-1}{2} \{(\sum_{j=1}^i j p^{s_1+j-1}) + p^i(m_{i+1} - 1)\}$, we gather that the inequality $\frac{|G/H_{i+1}|}{g_{C/H_{i+1}}} > \frac{2p}{p-1}$ is equivalent to the following condition on m_{i+1} :

$$m_{i+1} < p^{s_1+1} - \left(\sum_{j=1}^i j p^{s_1+j-1-i}\right) + 1 = p^{s_1}(p-1) + \sum_{j=2}^i (p-(j+1)) p^{s_1+j-i-1} + (p-1) p^{s_1-i} + 1 \quad (4)$$

We now assume that $\ell_{i,i+1}$ is the null linear form. Then, for all y in V , $\Delta_y(f_i) = \sum_{j=1}^{i-1} \ell_{j,i+1}(y) f_j(X) \pmod{\wp(k[X])}$. This ensures that the function field of the curve $\mathcal{C} : W_j^p - W_j = f_j(X)$, with $1 \leq j \leq i+1$ and $j \neq i$, is a Galois extension of $k(X)$ whose group \mathcal{H} is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^i$ and, as usual, the group of translations by V extends to an automorphism group of \mathcal{C} , say \mathcal{G} , with the following exact sequence:

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{G} \longrightarrow V \longrightarrow 0$$

We compute the quotient $\frac{|\mathcal{G}|}{g_c}$. As $|\mathcal{G}| = p^{s_1+i+1}$ and $g_c = \frac{p-1}{2} \{(\sum_{j=1}^{i-1} j p^{s_1+j-1}) + p^{i-1}(m_{i+1}-1)\}$, one can check that $\frac{|\mathcal{G}|}{g_c} > \frac{2p}{p-1}$ if and only if

$$m_{i+1} < p^{s_1+1} - \sum_{j=1}^{i-1} j p^{s_1+j-i} + 1 = p^{s_1}(p-1) + \sum_{j=1}^{i-2} (p-(j+1)) p^{s_1+j-i} + (p-1) p^{s_1-i+1} + 1 \quad (5)$$

The condition (5) is verified since it is implied by (4). It follows that $(\mathcal{C}, \mathcal{G})$ is a big action. By Theorem 3.13, it implies that the i -th function: f_{i+1} , lies in Σ_{i+1} , which contradicts the hypothesis $f_{i+1} \in \Sigma_{i+2} - \Sigma_{i+1}$. Therefore, $\ell_{i,i+1}$ is a nonzero linear form, which completes the induction and prove the equivalence between 1 and 2.

We now prove the last statement on the upper ramification filtration of G_2 . Starting from a given adapted basis of A : $\{\overline{f_1(X)}, \dots, \overline{f_n(X)}\}$, we get, by duality with respect to the Artin-Schreier pairing, a basis of G_2 , say $\{g_1, \dots, g_n\}$. As proved, for all i in $\{1, \dots, n\}$, $m_i = 1 + i p^{s_1}$, the jumps in the upper ramification of A , as defined in section 2.3.1, are: $\mu_i = m_{i+1} = 1 + (i+1)p^{s_1}$, for all i in $\{0, \dots, n-1\}$. Put $\mu_n := 1 + m_n$. Then, $A^{\mu_0} = \{0\}$ and, for all i in $\{1, \dots, n\}$, A^{μ_i} is the \mathbb{F}_p -subvector space of A generated by $\overline{f_1(X)}, \dots, \overline{f_i(X)}$. By duality (see Proposition 2.5), $(G_2)^{\nu_n} = (G_2)^{\mu_n} = \{e\} = \Lambda_0(G)$ and, for all i in $\{0, \dots, n-1\}$, $(G_2)^{\nu_i} = (G_2)^{\mu_i}$ is the \mathbb{F}_p -subvector space of G_2 generated by g_{i+1}, \dots, g_n , which is precisely $\Lambda_{n-i}(G)$, as seen in Proposition 5.2. \square

6 Examples.

We conclude this paper with some examples illustrating the special case of big actions described in Theorem 5.8, namely the big actions (C, G) with a p -elementary abelian G_2 such that each f_i lies in $\Sigma_{i+1} - \Sigma_i$. Note that Theorem 5.8 is twofold: on the one hand, it gives a group-theoretic characterization of G (cf. 5.8.2) and, on the other hand, it displays a dual point of view related to the parametrization of the cover (cf. 5.8.1). When studying the special family explicitly constructed via equations in Proposition 6.1, the second point of view naturally dominates in the proof. On the contrary, when exploring a universal family as in section 6.2, we are lead to combine both aspects.

Notation. The notations concerning big actions are those fixed in section 3.2. Moreover, let $W(k)$ be the ring of Witt vectors with coefficients in k . Then, for any $\sigma \in k$, we denote by $\tilde{\sigma}$ the Witt vector $\tilde{\sigma} := (\sigma, 0, 0, \dots) \in W(k)$. For any $S(X) := \sum_{i=0}^s \sigma_i X^i \in k[X]$, we denote by $\tilde{S}(X)$ the polynomial $\sum_{i=0}^s \tilde{\sigma}_i X^i \in W(k)[X]$.

6.1 A special family.

6.1.1 Case $s_1 = 1$.

Let $p \geq 3$ and $1 \leq n \leq p-1$. We first exhibit a special family of big actions (C, G) which satisfy the conditions of Theorem 5.8 with $s_1 = 1$ and so, $v = \dim_{\mathbb{F}_p} V = 2$. We shall distinguish the cases $n < p-1$ and $n = p-1$. When investigating the properties of the corresponding group G , we show, among others, that G is a capable group (see Definition 6.8) as studied by [Ha40] and [BT82].

Proposition 6.1. *Let $p \geq 3$. Let $S(X) := \wp(X)$ and $Q(X) := \wp(S(X))$. Call V the \mathbb{F}_p -vector space V consisting of the set of zeroes of the polynomial Q . Then, $V \simeq (\mathbb{Z}/p\mathbb{Z})^2$.*

1. Let n in $\{1, \dots, p-2\}$. For all i in $\{1, \dots, n\}$, we denote

$$g_i(X) := \frac{S(X)^{i+1}}{(i+1)!}$$

Let $f_i := \text{red}(g_i)$ be the reduced representative of g_i , as defined in the introduction. Let $C[n]$ be the curve parametrized by the n Artin-Schreier equations: $W_i^p - W_i = f_i(X)$, for $1 \leq i \leq n$. Let $K_n := k(C[n])$ be the function field of $C[n]$ and $H[n] \simeq (\mathbb{Z}/p\mathbb{Z})^n$ be the Galois group of $K_n/k(X)$. Then, the group of translations of the affine line: $\{\tau_y : X \rightarrow X + y, y \in V\}$ extends to an automorphism p -group of $C[n]$, say $G[n]$, such that we get the exact sequence:

$$0 \longrightarrow H[n] \simeq (\mathbb{Z}/p\mathbb{Z})^n \longrightarrow G[n] \longrightarrow V \simeq (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow 0$$

In this case, the pair $(C[n], G[n])$ is a big action with $G[n]_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$. Moreover, this big action satisfies the conditions of Theorem 5.8 with $s_1 = 1$.

2. We now study the case $n = p - 1$. We define $g_{p-1}(X) \in k[X]$ as the reduction mod p of the polynomial

$$\frac{1}{p!} ((X^p - X)^p - X^{p^2} + X^p) \in W(k)[X]$$

Let f_{p-1} be the reduced representative of g_{p-1} . Let $C[p-1]$ be the curve parametrized by the $p-1$ Artin-Schreier equations: $W_i^p - W_i = f_i(X)$, for $1 \leq i \leq p-1$, where the $p-2$ first f_i 's are defined as in the first case. Let $K_{p-1} := k(C[p-1])$ be the function field of $C[p-1]$ and $H[p-1] \simeq (\mathbb{Z}/p\mathbb{Z})^{p-1}$ be the Galois group of $K_{p-1}/k(X)$. Then, the group of translations of the affine line: $\{\tau_y : X \rightarrow X + y, y \in V\}$ extends to an automorphism p -group of $C[p-1]$, say $G[p-1]$, with the following exact sequence:

$$0 \longrightarrow H[p-1] \simeq (\mathbb{Z}/p\mathbb{Z})^{p-1} \longrightarrow G[p-1] \longrightarrow V \simeq (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow 0$$

In this case, the pair $(C[p-1], G[p-1])$ is a big action with $G[p-1]_2 \simeq (\mathbb{Z}/p\mathbb{Z})^{p-1}$. Moreover, this big action satisfies the conditions of Theorem 5.8 with $s_1 = 1$.

Proof: Using Proposition 2.11, we first observe that the second case implies the first one, when excluding the last equation: $W_{p-1}^p - W_{p-1} = f_{p-1}(X)$. Therefore, it is sufficient to prove the second point.

Fix $y \in V$. We begin by calculating $\Delta_y(g_i)$ for $1 \leq i \leq p-2$. So, take i in $\{1, \dots, p-2\}$. One first shows that

$$\Delta_y(g_i) = g_i(X+y) - g_i(X) = \sum_{j=1}^{i-1} \frac{S(y)^{i-j}}{(i-j)!} g_j(X) + g_i(y) + \frac{S(y)^i}{i!} S(X)$$

where the first sum is empty when $i = 1$. Since $S(y)$ lies in \mathbb{F}_p for all y in $Z(Q) = V$, one gets:

$$\Delta_y(g_i) = \sum_{j=1}^{i-1} \frac{S(y)^{i-j}}{(i-j)!} g_j(X) + g_i(y) + \wp\left(\frac{S(y)^i}{i!} X\right).$$

As k is an algebraically closed field, $g_i(y) = 0 \bmod \wp(k[X])$. We gather that $\Delta_y(g_1) = 0 \bmod \wp(k[X])$ and that, for all i in $\{2, \dots, p-2\}$, $\Delta_y(g_i) = \sum_{j=1}^{i-1} \ell_{j,i}(y) g_j(X) \bmod \wp(k[X])$ with $\ell_{j,i}(y) := \frac{S(y)^{i-j}}{(i-j)!} \in \mathbb{F}_p$. Since $g_i(X) - f_i(X)$ lies in $\wp(k[X])$ and since each $\ell_{j,i}(y)$ belongs to \mathbb{F}_p , it follows that

$$\forall i \in \{1, \dots, p-2\}, \quad \Delta_y(f_i) = \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X) \bmod \wp(k[X]) \quad \text{with} \quad \ell_{j,i}(y) := \frac{S(y)^{i-j}}{(i-j)!}$$

Now fix y in V and calculate $\Delta_y(g_{p-1})$. As $X^p - X = S(X) \bmod p$, we first notice that $(X^p - X)^p = \tilde{S}(X)^p \bmod p^2 W(k)[X]$. It follows that g_{p-1} can also be seen as the reduction mod p of the polynomial: $\frac{1}{p!} (\tilde{S}(X)^p - X^{p^2} + X^p) \in W(k)[X]$. By the same token, from $S(X+y) = S(X) + S(y) \bmod p$, we gather that $\tilde{S}(X + \tilde{y})^p = (\tilde{S}(X) + \tilde{S}(\tilde{y}))^p \bmod p^2 W(k)[X]$. It follows that

$$\tilde{S}(X + \tilde{y})^p - \tilde{S}(X)^p - \tilde{S}(\tilde{y})^p = \sum_{i=1}^{p-1} \binom{p}{i} \tilde{S}(X)^i \tilde{S}(\tilde{y})^{p-i} \bmod p^2 W(k)[X]$$

Likewise,

$$(X + \tilde{y})^p - X^p - \tilde{y}^p - (X + \tilde{y})^{p^2} + X^{p^2} + \tilde{y}^{p^2} = \sum_{i=1}^{p-1} \binom{p}{i} (X^i \tilde{y}^{p-i} - X^{pi} \tilde{y}^{p(p-i)}) \bmod p^2 W(k)[X]$$

Then, we obtain the following equalities :

$$\begin{aligned} & \frac{1}{p!} (\tilde{S}(X + \tilde{y})^p - \tilde{S}(X)^p - \tilde{S}(\tilde{y})^p + (X + \tilde{y})^p - X^p - \tilde{y}^p - (X + \tilde{y})^{p^2} + X^{p^2} + \tilde{y}^{p^2}) \\ &= \sum_{i=1}^{p-2} \frac{\tilde{S}(\tilde{y})^{p-i-1}}{(p-i-1)!} \frac{\tilde{S}(X)^{i+1}}{(i+1)!} + \frac{\tilde{S}(\tilde{y})^{p-1}}{(p-1)!} \tilde{S}(X) + \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p!} (X^i \tilde{y}^{p-i} - X^{pi} \tilde{y}^{p(p-i)}) \bmod pW(k)[X] \\ &= \sum_{i=1}^{p-2} \frac{S(y)^{p-i-1}}{(p-i-1)!} \frac{S(X)^{i+1}}{(i+1)!} + \frac{S(y)^{p-1}}{(p-1)!} S(X) + \sum_{i=1}^{p-1} \frac{(-1)^i}{i} (X^i y^{p-i} - X^{ip} y^{p(p-i)}) \bmod pW(k)[X] \end{aligned}$$

since the kernel of the map: $\begin{cases} W(k) \rightarrow k \\ (a_0, a_1, \dots) \rightarrow a_0 \end{cases}$ is $pW(k)$. From $S(y) \in \mathbb{F}_p$, we infer:

$$\Delta_y(g_{p-1}) = \sum_{i=1}^{p-2} \frac{S(y)^{p-i-1}}{(p-i-1)!} g_i(X) + \wp\left(\frac{S(y)^{p-1}}{(p-1)!} X\right) + \wp\left(\sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} X^i y^{p-i}\right) + g_{p-1}(y)$$

It follows that $\Delta_y(g_{p-1}) = \sum_{i=1}^{p-2} \ell_{i,p-1}(y) g_i(X) \pmod{\wp(k[X])}$, with $\ell_{i,p-1}(y) = \frac{S(y)^{p-1-i}}{(p-1-i)!} \in \mathbb{F}_p$. Since $g_i - f_i \in \wp(k[X])$ and $\ell_{i,p-1}(y) \in \mathbb{F}_p$,

$$\Delta_y(f_{p-1}) = \sum_{i=1}^{p-2} \ell_{i,p-1}(y) f_i(X) \pmod{\wp(k[X])} \quad \text{with} \quad \ell_{i,p-1}(y) = \frac{S(y)^{p-1-i}}{(p-1-i)!}$$

By Galois Theory, this ensures that the group $G[p-1]$ is well-defined. Furthermore, it is easy to check that for all i in $\{1, \dots, p-1\}$, $\deg f_i = 1 + ip$. In this case, the same computation as in the end of the proof of Theorem 5.6 shows that $\frac{|G[p-1]|}{|G[p-1]|} = \frac{2p}{p-1} \frac{p^{p-1}(p-1)^2}{(p-1)p^{p-1}(p-1)+1-p^{p-1}}$, which proves that the pair $(C[p-1], G[p-1])$ is a big action. To conclude, note that for all i in $\{1, \dots, p-2\}$ and for all y in V , $\ell_{i,i+1}(y) = S(y)$, which proves that $\ell_{i,i+1}$ is a nonzero linear form from V to \mathbb{F}_p . Therefore, because of Remarks 2.10 and 6.3, $G[p-1]$ satisfies the third assertion of Proposition 5.2 and then the conditions of Theorem 5.8. \square

Remark 6.2. *The preceding proof shows that, in the case of Proposition 6.1,*

$$\forall y \in V, \forall i \in \{2, \dots, p-1\} \quad \text{and} \quad \forall j \in \{1, \dots, i-1\}, \quad \ell_{j,i}(y) = \frac{S(y)^{i-j}}{(i-j)!}$$

It follows that the matrix $L(y)$ defined in section 2.4 reads: $L(y) = \exp(S(y)J) = \sum_{i=0}^{n-1} \frac{(S(y)J)^i}{i!}$ where J is the $n \times n$ nilpotent matrix:

$$J := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition 6.3. *Let $(C[n], G[n])$ be the big action described in the first point of Proposition 6.1, i.e. with $n < p-1$. The notations are those introduced in Proposition 6.1.*

1. *Let σ in $G[n]$. Let y in V such that $y := \sigma(X) - X$. Then,*

$$\sigma[W] = {}^t L(y)[W] + X[\mathcal{R}(y)] + [Z(y)]$$

where ${}^t L(y)$ denotes the transpose matrix of the upper triangular matrix $L(y)$ defined in section 2.4., $[W] := {}^t [W_1, \dots, W_n]$, $[\mathcal{R}(y)] := {}^t [\frac{S(y)}{1!}, \dots, \frac{S(y)^n}{n!}]$, and $[Z(y)] := {}^t [Z_1(y), \dots, Z_n(y)]$, where, for all i in $\{1, \dots, n\}$, $Z_i(y)$ is an element of k which satisfies $\wp(Z_i(y)) = g_i(y)$.

2. *The group $G[n]$ has exponent p .*

Proof:

1. For the need of the proof, it is more convenient to work with the non-reduced functions, namely the functions g_i 's. However, we still write the equations: $W_i^p - W_i = g_i(X)$, without changing the notation of W_i . As seen in the proof of Proposition 6.1,

$$\forall y \in V, \quad \forall i \in \{1, \dots, n\}, \quad \Delta_y(g_i) = \sum_{j=1}^{i-1} \ell_{j,i}(y) g_j(X) + g_i(y) + \wp(P_i(X, y))$$

where the sum on j is empty for $i = 1$ and where $P_i(X, y) := \frac{S(y)^i}{i!} X$. From $W_i^p - W_i = g_i(X)$, we infer that $\sigma(W_i^p - W_i) = \sigma(g_i(X)) = \Delta_y(g_i)$, which implies $\wp(\sigma(W_i)) = \wp(W_i) + \sum_{j=1}^{i-1} \ell_{j,i}(y) W_j + X \frac{S(y)^i}{i!} + g_i(y)$. Therefore, for all i in $\{1, \dots, n\}$,

$$\sigma(W_i) = W_i + \sum_{j=1}^{i-1} \ell_{j,i}(y) W_j + X \frac{S(y)^i}{i!} + Z_i(y)$$

where $Z_i(y)$ is an element of k such that $\wp(Z_i(y)) = g_i(y)$. Using the vector notations of the proposition, we thus obtain the expected formula.

2. To prove the second assertion, we compute $\sigma^p[W]$. An induction shows that:

$$\begin{aligned}\sigma^p[W] &= ({}^tL(y))^p[W] + X \left(\sum_{i=0}^{p-1} ({}^tL(y))^i [\mathcal{R}(y)] \right) \\ &\quad + y \left(\sum_{i=0}^{p-2} (p-i-1) ({}^tL(y))^i [\mathcal{R}(y)] + \sum_{i=0}^{p-1} ({}^tL(y))^i [Z(y)] \right)\end{aligned}$$

We first notice that $({}^tL(y))^p$ is equal to the identity matrix I , since ${}^tL(y) - I$ is nilpotent of size $n \leq p-2$. Moreover, by Remark 6.2, ${}^tL(y) = \exp(J(y)) = \sum_{i=0}^{n-1} \frac{J(y)^i}{i!}$, where $J(y) := S(y) {}^tJ$. Accordingly,

$$\begin{aligned}\sum_{i=0}^{p-1} ({}^tL(y))^i &= \sum_{i=0}^{p-1} \exp(iJ(y)) \\ &= I + \sum_{i=1}^{p-1} \sum_{j=0}^{n-1} \frac{(iJ(y))^j}{j!} = I + \sum_{j=0}^{n-1} \frac{J(y)^j}{j!} \sum_{i=1}^{p-1} i^j \\ &= I + \left(\sum_{i=1}^{p-1} i^0 \right) I + \sum_{j=1}^{n-1} \frac{J(y)^j}{j!} \sum_{i=1}^{p-1} i^j \\ &= \sum_{j=1}^{n-1} \frac{J(y)^j}{j!} \sum_{i=1}^{p-1} i^j \pmod{p}\end{aligned}$$

But one easily checks that $\mathcal{N}(j) := \sum_{i=1}^{p-1} i^j = 0 \pmod{p}$ for all j in $\{1, \dots, p-2\}$. Since $n-1 \leq p-3$, we gather that $\sum_{i=0}^{p-1} ({}^tL(y))^i = 0 \pmod{p}$.

To conclude, the last sum to compute is $\mathfrak{S} := \sum_{i=0}^{p-2} (p-i-1) ({}^tL(y))^i$. Likewise, one shows

$$\begin{aligned}\mathfrak{S} &= \sum_{i=0}^{p-2} (p-i-1) \exp(iJ(y)) \\ &= (p-1)I + \sum_{i=1}^{p-2} (p-i-1) \sum_{j=0}^{n-1} \frac{(iJ(y))^j}{j!} \\ &= (p-1)I - \sum_{i=1}^{p-2} (i+1)I - \sum_{j=1}^{n-1} \frac{J(y)^j}{j!} \left(\sum_{i=1}^{p-1} (i+1)i^j \right) \pmod{p} \\ &= (p-1)I - (\mathcal{N}(1) - 1)I - \sum_{j=1}^{n-1} \frac{J(y)^j}{j!} \mathcal{N}(j) - \sum_{j=2}^n \frac{J(y)^j}{j!} \mathcal{N}(j) \pmod{p}\end{aligned}$$

Since $\mathcal{N}(j) = 0$ when $1 \leq j \leq n \leq p-2$, it follows that $\sum_{i=0}^{p-2} (p-i-1) ({}^tL(y))^i = 0 \pmod{p}$. As $\sigma^p(X) = X + py = X \pmod{p}$, we gather that the order of σ divides p . Therefore, the group $G[n]$ has exponent p . \square

Proposition 6.4. *Let $(C[p-1], G[p-1])$ be the big action described in the second point of Proposition 6.1, i.e. with $n = p-1$. We keep the notations introduced in Proposition 6.1. For all y in k , we also define $T(X, y) := \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} X^i y^{p-i}$, i.e. the reduction mod p of $\frac{1}{p}\{(X + \tilde{y})^p - X^p - \tilde{y}^p\} \in W(k)[X]$.*

1. Let σ in $G[p-1]$. Let y in V such that $y := \sigma(X) - X$. Then,

$$\sigma[W] = {}^tL(y)[W] + X[\mathcal{R}(y)] + [Z(y)] + [T(X, y)]$$

where ${}^tL(y)$ denotes the transpose matrix of the matrix $L(y)$ defined in section 2.4,

$[W] := {}^t[W_1, \dots, W_{p-1}]$, $[\mathcal{R}(y)] := {}^t[\frac{S(y)}{1!}, \dots, \frac{S(y)^{p-1}}{(p-1)!}]$, $[T(X, y)] = {}^t[0, 0, \dots, 0, T(X, y)]$ and $[Z(y)] := {}^t[Z_1(y), \dots, Z_{p-1}(y)]$ where, for all i in $\{1, \dots, n\}$, $Z_i(y)$ is an element of k satisfying $\wp(Z_i(y)) = g_i(y)$.

2. Let $\sigma \in G[p-1]$ as in the first point. Then, if $y \neq 0$, σ has order p^2 . Otherwise, the order of σ divides p . In particular, the group $G[p-1]$ has exponent p^2 .

Proof:

1. The proof of the second point of Proposition 6.1 shows that

$$\forall y \in V, \quad \Delta_y(g_{p-1}) = \sum_{j=1}^{p-2} \ell_{j,p-1}(y) W_j + \wp(P_{p-1}(X, y)) + g_{p-1}(y)$$

where

$$P_{p-1}(X, y) := \frac{S(y)^{p-1}}{(p-1)!} X + \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} X^i y^{p-i} = \frac{S(y)^{p-1}}{(p-1)!} X + T(X, y)$$

The same calculation as in the proof of Proposition 6.3 yields:

$$\sigma(W_{p-1}) = W_{p-1} + \sum_{j=1}^{p-2} \ell_{j,p-1}(y) W_j + \frac{S(y)^{p-1}}{(p-1)!} X + T(X, y) + Z_{p-1}(y)$$

where $Z_{p-1}(y)$ is an element of k such that $\wp(Z_{p-1}(y)) = g_{p-1}(y)$. The formula of the first point then derives from the vector notation together with the expression of the others $\sigma(W_i)$, for $1 \leq i \leq p-2$, obtained in Proposition 6.3.

2. We now calculate $\sigma^p[W]$. As in the previous proof, an induction shows that:

$$\begin{aligned} \sigma^p[W] &= ({}^tL(y))^p[W] + X \left(\sum_{i=0}^{p-1} ({}^tL(y))^i [\mathcal{R}(y)] \right) + y \left(\sum_{i=0}^{p-2} (p-i-1) ({}^tL(y))^i [\mathcal{R}(y)] \right) \\ &\quad + \left(\sum_{i=0}^{p-1} ({}^tL(y))^i [Z(y)] \right) + \sum_{i=0}^{p-1} [T(X + i y, y)] \end{aligned}$$

Still as in the proof of Proposition 6.3, ${}^tL(y)^p = I$ and $\sum_{i=0}^{p-1} ({}^tL(y))^i = \sum_{j=1}^{p-2} \frac{J(y)^j}{j!} \mathcal{N}(j)$, where $\mathcal{N}(j) := \sum_{i=1}^{p-1} i^j = 0 \pmod p$, for j in $\{1, \dots, p-2\}$. Besides, as previously seen,

$$\begin{aligned} \sum_{i=0}^{p-2} (p-i-1) ({}^tL(y))^i &= - \sum_{j=1}^{p-2} \frac{J(y)^j}{j!} \{ \mathcal{N}(j) + \mathcal{N}(j+1) \} \\ &= - \frac{J(y)^{p-2}}{(p-2)!} \mathcal{N}(p-1) = \frac{J(y)^{p-2}}{(p-2)!} = J(y)^{p-2} \pmod p \end{aligned}$$

Then, $y \left(\sum_{i=0}^{p-2} (p-i-1) ({}^tL(y))^i [\mathcal{R}(y)] \right) = y (S(y)^t J)^{p-2} [\mathcal{R}(y)] = {}^t [0, \dots, 0, y S(y)^{p-1}] \pmod p$. To complete the calculation, one has to compute $\sum_{i=0}^{p-1} T(X + i y, y)$. As T is the reduction mod p of the polynomial $\frac{1}{p} \{ (X + \tilde{y})^p - X^p - \tilde{y}^p \}$, it follows that

$$\begin{aligned} \sum_{i=0}^{p-1} T(X + i y, y) &= \frac{1}{p} \sum_{i=0}^{p-1} \{ (X + (1+i) \tilde{y})^p - (X + i \tilde{y})^p - \tilde{y}^p \} \pmod p \\ &= \frac{1}{p} \{ (X + p \tilde{y})^p - X^p - p \tilde{y}^p \} = -y^p \pmod p \end{aligned}$$

Therefore, $\sigma[W]^p = [W] + {}^t [0, 0, 0, \dots, 0, y S(y)^{p-1} - y^p] \pmod p$. If $y \in \mathbb{F}_p$, or equivalently $S(y) = 0$, then $\sigma[W]^p = [W] + {}^t [0, 0, 0, \dots, 0, y] \pmod p$. Otherwise, $S(y) \neq 0$ and, as $S(y)$ lies in \mathbb{F}_p , $S(y)^{p-1} = 1 \pmod p$, which implies $\sigma[W]^p = [W] + {}^t [0, 0, 0, \dots, 0, y - y^p] = [W] + {}^t [0, 0, 0, \dots, -S(y)] \pmod p$. We gather that if $y = 0$, the order of σ divides p . Otherwise, σ has order p^2 . This proves the second assertion. \square

Contrary to the preceding propositions, the next result describing the center of the group $G[n]$ is common to both cases of Proposition 6.1, i.e. $n < p-1$ and $n = p-1$.

Proposition 6.5. *Let $(C[n], G[n])$ be a big action as described in the first or the second point of Proposition 6.1, i.e. with $n < p-1$ or $n = p-1$. Let σ in $G[n]$. Then, σ belongs to the center of $G[n]$ if and only if*

$$\sigma(X) = X \quad \text{and} \quad \forall i \in \{1, \dots, n-1\}, \sigma(W_i) = W_i$$

It follows that the center of $G[n]$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Proof: Throughout the proof, we keep the notations introduced in Proposition 6.3 and 6.4.

1. We first focus on the case $n < p-1$. Let σ in $Z(G[n])$ and y in V such that $y := \sigma(X) - X$. By definition of $\phi(y)$, for all g in $G[n]_2$, $\phi(y)(g) = \sigma^{-1} g \sigma = g$, since σ lies in the center of $G[n]$. So $\phi(y) = id$ and $L(y)$ is the identity matrix. It follows from Remark 6.2 that $S(y) = \ell_{1,2}(y) = 0$. Then, $g_i(y) = 0$, for all $1 \leq i \leq n$, and $Z_i(y)$, which satisfies $\wp(Z_i(y)) = g_i(y)$, lies in \mathbb{F}_p . So, by Proposition 6.3, $\sigma[W] = {}^t L(y)[W] + X[\mathcal{R}(y)] + [Z(y)] = [W] + [Z(y)]$. We now choose some

u in V such that $S(u) \neq 0$ and consider τ in $G[n]$ such that $\tau(X) = X + u$. Then, still by Proposition 6.3, $\tau[W] = {}^t L(u)[W] + X[\mathcal{R}(u)] + [Z(u)]$. One checks that $\sigma\tau[W] = \tau\sigma[W]$ if and only if $y[\mathcal{R}(u)] + ({}^t L(u) - I)[Z(y)] = 0$. As $S(u) \neq 0$, it implies that $y = 0$ and $Z_i(y) = 0$ for all i in $\{1, \dots, n-1\}$.

Conversely, if $y = 0$, then $S(y) = 0$. As above, it implies $Z_i(y) \in \mathbb{F}_p$, for $1 \leq i \leq n$, and $L(y) = I$. It follows that $\sigma[W] = [W] + [Z(y)]$. Consider τ in $G[n]$ and u in V such that $u := \tau(X) - X$. On the one hand, $\sigma\tau(X) = X + u + y = \tau\sigma(X)$. On the other hand, as seen above, $\sigma\tau[W] = \tau\sigma[W]$ if and only if $y[\mathcal{R}(u)] + ({}^t L(u) - I)[Z(y)] = 0$. If $S(u) = 0$, this equality is trivially true. Otherwise, it comes from $Z_i(y) = 0$ for all i in $\{1, \dots, n-1\}$. Therefore, σ lies in the center of $G[n]$.

As a conclusion, σ belongs to the center of $G[n]$ if and only if $\sigma(X) = X$, $\sigma(W_i) = W_i$, with $1 \leq i \leq n-1$, and $\sigma(W_n) = W_n + Z_n$, with Z_n in \mathbb{F}_p . Thus, $Z(G[n]) \simeq \mathbb{Z}/p\mathbb{Z}$.

2. When $n = p-1$, the result is the same but the proof is slightly different. As above, we first notice that, if σ lies in $Z(G[n])$, $L(y) = I$, $S(y) = 0$ and then y is in \mathbb{F}_p . Write $\tilde{y}^p = \tilde{y} + pR$ with R in $W(k)$. It implies that $g_{p-1}(y)$, which is the reduction mod p of $\frac{1}{p!}(\{\tilde{y}^p - \tilde{y}\}^p - \tilde{y}^{p^2} + \tilde{y}^p)$, is zero. Accordingly, $Z_{p-1}(y)$ also lies in \mathbb{F}_p . Then, by Proposition 6.4, $\sigma[W] = {}^t L(y)[W] + X[\mathcal{R}(y)] + [Z(y)] + [\mathcal{T}(X, y)] = [W] + [Z(y)] + [\mathcal{T}(X, y)]$. We now choose some u in V such that $S(u) \neq 0$ and consider τ in $G[n]$ such that $\tau(X) = X + u$. Still by Proposition 6.4, $\tau[W] = {}^t L(u)[W] + X[\mathcal{R}(u)] + [Z(u)] + [\mathcal{T}(X, u)]$. One checks that $\sigma\tau[W] = \tau\sigma[W]$ if and only if

$$y[\mathcal{R}(u)] + ({}^t L(u) - I)[Z(y)] + {}^t L(u)[\mathcal{T}(X, y)] + [\mathcal{T}(X + y, u)] - [\mathcal{T}(X, u)] - [\mathcal{T}(X + u, y)] = 0 \quad (6)$$

As $S(u) \neq 0$, it implies that $y = 0$. Then, $\mathcal{T}(X, y) = \mathcal{T}(X + u, y) = 0$ and $\mathcal{T}(X + y, u) = \mathcal{T}(X, u)$. Thus, one gets: $({}^t L(u) - I)[Z(y)] = 0$, which implies $Z_i(y) = 0$ for all i in $\{1, \dots, n-1\}$.

Conversely, if $y = 0$, then $S(y) = 0$, $L(y) = I$ and $\mathcal{T}(X, y) = 0$. It follows that $\sigma[W] = [W] + [Z(y)]$. Consider τ in $G[n]$ and u in V such that $u := \tau(X) - X$. On the one hand, $\sigma\tau(X) = X + u + y = \tau\sigma(X)$. On the other hand, as seen above, $\sigma\tau[W] = \tau\sigma[W]$ if and only if (6) is satisfied. If $S(u) = 0$, this equality is trivially true. Otherwise, it comes from $Z_i(y) = 0$ for all i in $\{1, \dots, n-1\}$ together with $\mathcal{T}(X, y) = \mathcal{T}(X + u, y) = 0$ and $\mathcal{T}(X + y, u) = \mathcal{T}(X, u)$ obtained because $y = 0$. Therefore, σ lies in the center of $G[n]$ and we conclude as in the previous case. \square

Corollary 6.6. *Let $p \geq 3$ and $2 \leq n \leq p-1$. We keep the notation of Proposition 6.1.*

1. *The group $G[1]$ is the extraspecial group of order p^3 and exponent p , namely the unique non abelian group of order p^3 and exponent p . Moreover, we get the following exact sequence:*

$$0 \longrightarrow Z(G[1]) \simeq \mathbb{Z}/p\mathbb{Z} \longrightarrow G[1] \longrightarrow (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow 0$$

2. *We also have the following exact sequence:*

$$0 \longrightarrow Z(G[n]) \simeq \mathbb{Z}/p\mathbb{Z} \longrightarrow G[n] \longrightarrow G[n-1] \longrightarrow 0$$

Proof:

1. The first assertion derives from [LM05] (Prop. 8.1).
2. As in Proposition 6.1, we call $K_n := k(C[n]) = k(X, W_1, \dots, W_n)$ the function field of the curve $C[n]$. Put $k(T) := K_n^{G[n]}$, where $T = Q(X)$, Q being defined as in Proposition 6.1. Then, Galois theory, combined with Proposition 6.5, gives the following exact sequence:

$$0 \longrightarrow \text{Gal}(K_n/K_{n-1}) \simeq Z(G[n]) \simeq \mathbb{Z}/p\mathbb{Z} \longrightarrow \text{Gal}(K_n/k(T)) \longrightarrow \text{Gal}(K_{n-1}/k(T)) \longrightarrow 0$$

The claim directly follows. \square

Remark 6.7. *Computation using MAGMA package on finite groups shows that, for $n \geq 2$, the group $G[n]$ is, in general, not uniquely determined by the group extension conditions mentionned in Corollary 6.6.*

Definition 6.8. *Following [Ha40] and [BT82], we say that a group G is capable if there exists a group Γ such that $G \simeq \frac{\Gamma}{Z(\Gamma)}$.*

We deduce from Corollary 6.6 the following:

Corollary 6.9. *Let $p \geq 3$ and $2 \leq n \leq p - 1$.*

1. *The group $G[n - 1]$ is capable, with $\Gamma = G[n]$.*
2. *In particular, the extraspecial group of order p^3 and exponent p , with $p > 2$, is capable with $\Gamma = G[2]$, a group of order p^4 .*

Remark 6.10. *Note that [BT82] (Example 1.12 p.187) gives another proof of the second statement of Corollary 6.9. Nevertheless, this proof uses some group Γ of order p^5 .*

6.1.2 General case.

Starting from the big actions defined in Proposition 6.1, for which $s_1 = 1$, we use the base change displayed in [MR08] (section 3) to obtain new ones which still satisfy the conditions of Theorem 5.8 but have arbitrary large s_1 .

Proposition 6.11. *Let $p \geq 3$, $1 \leq n \leq p - 1$ and $s_0 \in \mathbb{N}$. Let $S_0(X)$ be an additive separable polynomial of $k[X]$ with degree p^{s_0} . Let $(C[n], G[n])$ be the big action defined in Proposition 6.1. Consider the additive polynomial map $S_0 : \mathbb{P}_k^1 \rightarrow C[n]/G[n]_2 \simeq \mathbb{P}_k^1$.*

1. *Let $\tilde{C}[n] := C[n] \times_{\mathbb{P}_k^1} \mathbb{P}_k^1$ be the curve obtained after the base change defined by S_0 . Then, the cover $\tilde{C}[n] \rightarrow C[n]/G[n]$ is Galois with group $\tilde{G}[n] \simeq G[n] \times (\mathbb{Z}/p\mathbb{Z})^{s_0}$. Moreover, the pair $(\tilde{C}[n], \tilde{G}[n])$ is a big action with $\tilde{G}[n]_2 \simeq G[n]_2 \times \{0\}$ and $Z(\tilde{G}[n]) \simeq (\mathbb{Z}/p\mathbb{Z})^{s_0+1}$.*
2. *This big action $(\tilde{C}[n], \tilde{G}[n])$ satisfies the conditions of Theorem 5.8 with $s_1 = s_0 + 1$.*

Proof:

1. The first assertion derives from [MR08] (Prop. 3.1). Another proof consists in replacing X with $S_0(X)$ in the proof of Proposition 6.1, knowing that the calculation only requires S_0 to be additive.
2. One deduces from Lemma 3.7.2 and Lemma 3.7.7 that $f_i(X) \in \Sigma_{i+1} - \Sigma_i$ implies $f_i(S_0(X)) \in \Sigma_{i+1} - \Sigma_i$. The claim follows. Another proof consists in considering the filtration $(\Lambda_i(G[n]))_{i \geq 0}$, as defined in section 5.1. By Proposition 6.1, this filtration satisfies the first condition of Proposition 5.2. Then, one concludes by checking that, for all $i \geq 0$, $\Lambda_i(\tilde{G}[n]) \simeq \Lambda_i(G[n])$. \square

6.2 A universal family.

Under the hypotheses of Theorem 5.8, one already knows the form of the functions f_i 's, namely their degree $m_i = 1 + i p^{s_1}$ and their belonging to $\Sigma_{i+1} - \Sigma_i$. For given p , s_1 and $n \leq p - 1$, this naturally yields an algorithmic method to parametrize the functions f_i 's. In this way, we obtain a universal family parametrizing the big actions (C, G) that satisfy Theorem 5.8 with f_1 monic and $s_1 = 1$. Eventhough it theoretically works for any $p \geq 3$, in what follows, we merely illustrate this method in the special case $p = 5$ and $n \leq p - 1 = 4$. In this case, we also describe the corresponding space of parameters and, when $n = 2$, we give necessary and sufficient conditions on the parameters for two curves of the family to be isomorphic. We eventually characterize the subfamily corresponding to the special curves that are studied in section 6.1.1. Throughout this section, the notations concerning big actions are still those fixed in section 3.2.

Proposition 6.12. *Fix $p = 5$. Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $2 \leq n \leq p - 1$. We suppose that $s_1 = 1$. We also assume that (C, G) satisfies the conditions of Theorem 5.8. Then, there exists a coordinate X for the projective line $C/G_2 \simeq \mathbb{P}_1$ and an adapted basis for A as follows:*

For $n = 2$:

$$f_1(X) = X^6 + 2 \frac{b_0^{24} + 1}{b_0^4} X^2$$

$$f_2(X) = b_0^5 X^{11} + 4 b_0^{25} X^7 + 3 \frac{4 b_0^{48} + 1}{b_0^3} X^3 + b_5 X$$

Therefore, the parametrization of the functions f_i 's requires two algebraically independent parameters, namely b_0 and b_5 in k , with $b_0 \neq 0$.

For $n = 3$:

$$f_1(X) = X^6 + 2 \frac{b_0^{24}+1}{b_0^4} X^2$$

$$f_2(X) = b_0^5 X^{11} + 4 b_0^{25} X^7 + 3 \frac{4b_0^{48}+1}{b_0^3} X^3 + 2 \frac{c_7-c_7^5}{b_0^5} X$$

$$f_3(X) = 4 b_0^{10} X^{16} + 4 b_0^{30} X^{12} + 4 b_0^{50} X^8 + c_7^5 X^6 + 4 \frac{b_0^{72}+1}{b_0^2} X^4 + 2 c_7 \frac{c_7^4 b_0^{24}+1}{b_0} X^2 + c_9 X$$

Thus, the parametrization of the functions f_i 's requires three algebraically independent parameters, namely b_0 , c_7 and c_9 in k , with $b_0 \neq 0$.

For $n = 4$:

$$f_1(X) = X^6 + 2 \frac{b_0^{24}+1}{b_0^4} X^2$$

$$f_2(X) = b_0^5 X^{11} + 4 b_0^{25} X^7 + 3 \frac{4b_0^{48}+1}{b_0^3} X^3 + 2 \frac{c_7-c_7^5}{b_0^5} X$$

$$f_3(X) = 4 b_0^{10} X^{16} + 4 b_0^{30} X^{12} + 4 b_0^{50} X^8 + c_7^5 X^6 + 4 \frac{b_0^{72}+1}{b_0^2} X^4 + 2 c_7 \frac{c_7^4 b_0^{24}+1}{b_0} X^2 + 2 \frac{d_{11}-d_{11}^5}{b_0^3} X$$

$$f_4(X) = 2 b_0^{15} X^{21} + b_0^{35} X^{17} + 4 b_0^{55} X^{13} + d_8^5 b_0^5 X^{11} + 3 b_0^{75} X^9 + (4 d_8^{25} b_0^{25} + 4 b_0^{25} c_7^5 + b_0^{25} c_7^{25}) X^7 \\ + d_{11}^5 X^6 + (\frac{b_0^{24}+b_0^{48}}{b_0^3} c_7^{25} + \frac{2+4 b_0^{24}+4 b_0^{48}}{b_0^3} c_7^5 + \frac{3 c_7}{b_0^3} + \frac{4 b_0^{48}+4 b_0^{24}}{b_0^3} d_8^{25} + \frac{b_0^{24}+3+3 b_0^{48}}{b_0^3} d_8^5) X^3 \\ + 2 \frac{d_{11} (d_{11}^4 b_0^{24}+1)}{b_0^5} X^2 + d_{13} X$$

with

$$b_0^{96} = 1 \quad \text{and} \quad 2t + (3 b_0^{24} + 3)t^5 + 2 b_0^{24} t^{25} = 0 \quad \text{where} \quad t := d_8 - c_7$$

Accordingly, the parametrization of the functions f_i 's requires three algebraically independent parameters, namely c_7 , d_{11} and d_{13} in k .

Proof: We recall that, after an homothety and a translation, one can rigidify the parametrization and fix a coordinate X for the projective line $C/G_2 \simeq \mathbb{P}_1$ such that f_1 is a monic polynomial with no monomial of degree one. Furthermore, for $n \geq 3$, one also rigidify the functions f_i 's by assuming, following Proposition 5.4, that $\ell_{i,i+1} = \ell_{1,2}$. Thus, keeping the writing of exponents in 5-adic expansion, we write the functions f_i 's as follows:

$$f_1(X) = X^{1+5} + a X^2$$

$$f_2(X) = b_0^5 X^{1+2.5} + b_1 X^{2+5} + b_2 X^3 + b_3 X^{1+5} + b_4 X^2 + b_5 X$$

$$f_3(X) = c_0 X^{1+3.5} + c_1 X^{2+2.5} + c_2 X^{3+5} + c_3 X^4 + c_4^5 X^{1+2.5} + c_5 X^{2+5}$$

$$+ c_6 X^3 + c_7 X^{1+5} + c_8 X^2 + c_9 X$$

$$f_4(X) = d_0 X^{1+4.5} + d_1 X^{2+3.5} + d_2 X^{3+2.5} + d_3 X^{4+5} + b_0^{10} d_4 X^{1+3.5} + d_5 X^{2+2.5} + d_6 X^{3+5}$$

$$+ d_7 X^4 + b_0^5 d_8^5 X^{1+2.5} + d_9 X^{2+5} + d_{10} X^3 + d_{11}^5 X^{1+5} + d_{12} X^2 + d_{13} X$$

with $b_0 \neq 0$, $c_0 \neq 0$ and $d_0 \neq 0$. Note that, for convenience of calculation, some coefficients are directly written as p -powers. Following Proposition 2.13, we first calculate $Ad_{f_1}(Y) = Y^{25} + 2 a^5 Y^5 + Y$. As V is included in $Z(Ad_{f_1})$ and as, in our case, these two vector spaces have the same dimension over \mathbb{F}_p , namely $s_1 + 1 = 2 = 2 s_1$, we gather that $V = Z(Ad_{f_1})$. We now focus on the relation:

$$\forall y \in V, \quad \Delta_y(f_2) = \ell_{1,2}(y) f_1(X) \quad \text{mod } \wp(k[X]) \quad (7)$$

Computations using Maple show that for all y in V , $\ell_{1,2}(y) = 2 b_1 y + 2 b_0^5 y^5$. As $V = Z(Ad_{f_1})$, we deduce from Proposition 2.9 that $Ad_{f_1}(X)$ divides the polynomial $(2 b_1 X + 2 b_0^5 X^5)^5 - (2 b_1 X +$

$2b_0^5 X^5$). This requires: $b_1 = 4b_0^{25}$ and $a = 2\frac{b_0^{25}+1}{b_0^4}$. In addition, (7) also yields $b_2 = 3\frac{4b_0^{25}+1}{b_0^3}$ and $b_3 \in \mathbb{F}_5$. Accordingly, by replacing f_2 with $f_2 - b_3 f_1$, one can assume that $b_3 = 0$. It follows that $b_4 = 0$. We eventually obtain the expected expression of the functions f_1 and f_2 .

Likewise, the case $n = 3$ (resp. $n = 4$) is solved by studying the relation:

$$\forall y \in V, \quad \Delta_y(f_3) = \ell_{1,2}(y) f_2(X) + \ell_{1,3}(y) f_1(X) \quad \text{mod } \wp(k[X])$$

$$(\text{resp. } \forall y \in V, \quad \Delta_y(f_4) = \ell_{1,2}(y) f_3(X) + \ell_{2,4}(y) f_2(X) + \ell_{1,4}(y) f_1(X) \quad \text{mod } \wp(k[X])) \square$$

Remark 6.13. Keeping the notations of Proposition 6.12 with $n = 2$, one can show using [LM05] (Prop. 3.3) that the two pairs of parameters (b_0, b_5) and (b'_0, b'_5) give isomorphic k -curves C if and only if

$$\left(\frac{b'_0}{b_0}\right)^{24} = 1 \quad \text{and} \quad b'_5 = \pm \frac{b'_0}{b_0} b_5$$

We now emphasize the link with the special family studied in section 6.1. This allows us to characterize the group G , at least for $p = 5$ and $n < 4$.

Remark 6.14. 1. Following Proposition 6.11, we apply the linear base change: $X \rightarrow \lambda X$, with $\lambda \in k^\times$, to the big action defined in Proposition 6.1, for $n \geq 2$. Then, one finds a subfamily of the universal family displayed in Proposition 6.12 if and only if $\lambda \in \mathbb{F}_{25}^\times$. For instance, in the case $n = 2$, the subfamily obtained in this case is the one characterized by $b_0^{24} = 1$ and $b_5 = 0$.

2. We notice that the spaces of parameters of the universal family described in Proposition 6.12 when $n < 4$, are Zariski opens of linear affine spaces, which implies that they are irreducible and so connected. It follows from the preceding point and from Proposition 6.11 (with $s_0 = 0$) that the group G mentioned in Proposition 6.12 is isomorphic to the one obtained in Proposition 6.1. But, for $n = 4$, the space of parameters is no more connected (cf. $b_0^{96} = 1$), so the question remains open.

For given p and $2 \leq n < p - 1$, one does not know the connected components of the space of parameters. Nevertheless, the group structure can be approached via the proposition below that generalizes Proposition 6.5 and Corollary 6.6.

Proposition 6.15. Let $p \geq 3$ and $2 \leq n \leq p - 1$. Let (C, G) be a big action which satisfy the conditions of Theorem 5.8 with $s_1 = 1$.

1. The center of G , $Z(G)$, is included in its derived subgroup $D(G)$. It follows that $Z(G)$ is cyclic of order p .
2. Moreover, for all i in $\{1, \dots, n\}$, the quotient group $G/\Lambda_i(G)$ is capable.

Proof:

1. As G satisfies the conditions of Theorem 5.8, $\Lambda_{n-1}(G)$ is an index p subgroup of $G_2 = D(G)$. As $\Lambda_{n-1}(G) = (G_2)^{v_1}$ (cf. Theorem 5.8), the quotient curve $C/\Lambda_{n-1}(G)$ is the p -cyclic cover of the affine line parametrized by $W_1^p - W_1 = f_1(X)$. Since $v = s_1 + 1 = 2$, it follows from [LM05] that the group $G/\Lambda_{n-1}(G)$ is the extraspecial group of order p^3 and exponent p . In particular, its center is a p -cyclic group generated by τ such that $\tau(X) = X$ and $\tau(W_1) = W_1 + 1$. Now, take $\sigma \in Z(G)$. Then, σ induces $\tilde{\sigma} \in Z(G/\Lambda_{n-1}(G))$. So, $\sigma(X) = X$. As $k(X) = L^{D(G)}$, it implies that $Z(G)$ is included in $D(G)$. Besides, by Theorem 5.8, $\Lambda_1(G) = Z(G) \cap D(G) = Z(G) \simeq \mathbb{Z}/p\mathbb{Z}$, which proves that $Z(G)$ is p -cyclic.
2. Theorem 5.8 implies that the cover $C \rightarrow C/G_2$ is parametrized by n Artin-Schreier equations: $W_j^p - W_j = f_j(X) \in \Sigma_{j+1} - \Sigma_j$, with $1 \leq j \leq n$. Take i in $\{0, \dots, n-1\}$. Then, the curve $C/\Lambda_i(G)$ is parametrized by the $n-i$ first equations: $W_j^p - W_j = f_j(X) \in \Sigma_{j+1} - \Sigma_j$, with $1 \leq j \leq n-i$. It follows that the pair $(C/\Lambda_i(G), G/\Lambda_i(G))$ is a big action (cf. [MR08] Lemma 2.4) which still satisfies Theorem 5.8 with $s_1 = 1$. We deduce from the first point that $\Lambda_1(G/\Lambda_i(G)) = Z(G/\Lambda_i(G))$. As $\frac{G/\Lambda_i(G)}{\Lambda_1(G/\Lambda_i(G))} \simeq G/\Lambda_{i+1}(G)$, we get the exact sequence:

$$0 \longrightarrow Z(G/\Lambda_i(G)) \longrightarrow G/\Lambda_i(G) \longrightarrow G/\Lambda_{i+1}(G) \longrightarrow 0$$

The claim follows. \square

We conclude with the following

Problems:

1. For any p , find equations for the universal family (at least for $s_1 = 1$) as we obtained for the special family.
2. Compare the universal family corresponding to a given s_1 with the one obtained after a base change by a generic and additive polynomial map, applied to the universal family with $s_1 = 1$.

A last interesting question is raised by the following

Remark 6.16. *Proposition 6.12 seems to suggest that any p -cyclic étale cover of the affine line given by*

$$W_1^p - W_1 = f_1(X) := X S(X) \quad \text{with} \quad S \in k\{F\}$$

could be embedded in a big action (C, G) where C is parametrized by n Artin-Schreier equations:

$$W_i^p - W_i = f_i(X) \in \Sigma_{i+1} - \Sigma_i \quad \text{with} \quad 1 \leq i \leq n < p - 1$$

and that without any restriction on the coefficients of f_1 . Nevertheless, it is no more true for $n = p - 1$ unless the coefficients of $S(X)$ satisfy a specific algebraic condition to be determined (see e.g. $b_0^{96} = 1$ in Proposition 6.12).

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Magali ROCHER

Institut de Mathématiques de Bordeaux, Université de Bordeaux I, 351 cours de la Libération,
33405 Talence Cedex, France.

e-mail : `Magali.Rocher@math.u-bordeaux1.fr`